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ON OPTIMALITY OF THE LR TESTS IN THE SENSE OF EXACT SLOPES

Part II. Application to Individual Distributions

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Optimality of the likelihood ratio test statistic in the sense of exact slopes is established in the case, when sampling is made from q populations, and each of them is supposed to have either normal, or exponential, or Laplace, or Poisson distribution.

The optimality mentioned in the title will be proved by means of the assumptions (A I) – (A VII) or (AR II), (AR VI), which are presented in Section 1 of [12], where also a general framework and basic notations can be found. Since this paper is a continuation of [12], its sections are numbered from 3 to 6, and references to relations from Sections 1 and 2 are related to [12].

3. OPTIMALITY IN THE CASE OF THE NORMAL DISTRIBUTION

Let $k > 1$ be an integer. For a vector $\rho = (\rho_{12}, \dots, \rho_{1k}, \dots, \rho_{k-1,k})'$ belonging to R^a, $a = (k-1)k/2$, denote

$$\mathbf{R}(\rho) = \begin{pmatrix} 1, & \rho_{12}, & \dots & \rho_{1k} \\ \rho_{12}, & 1, & \dots & \rho_{2k} \\ \vdots & & & \vdots \\ \rho_{1k}, & \rho_{2k}, & \dots & 1 \end{pmatrix} \tag{3.1}$$

the symmetric matrix, having for $j > i$ on its (i,j) position the element ρ_{1k}, and put

(3.2)

$$\Xi = \{(\mu', \sigma_1, \dots, \sigma_k, \rho')' \in R^m;\ \sigma_i > 0 \textit{ for all } i,\ \mu \in R^k, \rho \in R^a,\ \mathbf{R}(\rho) \textit{ is positive definite}\},$$

where $m = 2k + (k-1)k/2$. If $\gamma = (\mu', \sigma_1, \dots, \sigma_k, \rho')'$ belongs to Ξ and

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sigma_k \end{pmatrix}, \qquad \mathbf{V}(\gamma) = \boldsymbol{\sigma}\mathbf{R}(\rho)\boldsymbol{\sigma} \tag{3.3}$$

then denote

$$\overline{P}_\gamma = N(\mu, \mathbf{V}(\gamma))$$

the k-dimensional normal distribution with mean μ and the covariance matrix $\mathbf{V}(\gamma)$. Thus the densities (1.14) are in this case

$$f(x,\gamma) = (2\pi)^{-k/2}|\mathbf{V}(\gamma)|^{-1/2}\exp[-\frac{1}{2}(x-\mu)'\mathbf{V}(\gamma)^{-1}(x-\mu)]\,, \tag{3.5}$$

and the dominating measure is the Lebesgue measure on $(R^k, \mathcal{B}^k)$.

Theorem 3.1. *Let (A I) hold.*

(I) If we denote for γ, $\gamma^* \in \Xi$ by

$$\tau(\gamma,\gamma^*) = [\sum_i (\gamma_i - \gamma_i^*)^2]^{1/2} \tag{3.6}$$

the usual Euclidean distance, then with the notation (3.1) – (3.5) the assumptions (AR II), (A III) – (A V), (AR VI), (A VII) are fulfilled.

(II) Let us assume that (1.4) holds, T_u is the statistic (1.28) and $\theta \in \Omega_1 - \Omega_0$. The relation (1.29) holds a.e. P_θ with $C(\theta) = 2J(\theta)$ (i.e., T_u is optimal in the sense of exact slopes), and if $\theta \in \Omega_1 - \overline{\Omega}_0$, then the assertions (I) and (II) of Theorem 1.3 are true.

First we recall that if the random variable χ^2_d has chi-square distribution with d degrees of freedom, then (as proved in a general case by means of Markov's inequality in [2], p. 2)

$$P[\,\chi^2_d \ge c\,] \le r^+_d(c)\,, \qquad P[\,\chi^2_d \le c\,] \le r^-_d(c)\,, \tag{3.7}$$

where

$$\begin{aligned} r^+_d(c) &= \inf\Big\{E\Big[\exp(t\chi^2_d)\Big]\exp(-tc);\; t \ge 0\Big\}\,,\\ r^-_d(c) &= \inf\Big\{E\Big[\exp(t\chi^2_d)\Big]\exp(-tc);\; t \le 0\Big\}\,. \end{aligned} \tag{3.8}$$

Since

$$E\Big[\exp(t\chi^2_d)\Big] = \begin{cases} (1-2t)^{-d/2} & t < 0.5\,,\\ +\infty & t \ge 0.5\,, \end{cases} \tag{3.9}$$

one can easily prove the following assertion.

Lemma 3.1.

$$\lim_{c\to\infty}\limsup_{\Delta\to+\infty} \frac{1}{\Delta}\log r^+_d(\Delta c) = -\infty\,, \tag{3.10}$$

$$\lim_{\Delta\to\infty}\limsup_{d\to+\infty} \frac{1}{d}\log r^+_d(\Delta d) = -\infty\,, \tag{3.11}$$

$$\lim_{\Delta\to 0+}\limsup_{d\to+\infty} \frac{1}{d}\log r^-_d(\Delta d) = -\infty\,. \tag{3.12}$$

P r o o f o f T h e o r e m 3.1. It is obvious from Theorem 1.2 that it is sufficient to prove the assertion (I).

(AR II) This assumption is obviously true.

(A III) In proving this assumption we shall use the notation

$$x^{(n)} = (x_1, \dots, x_n) \in X^n, \quad X = R^k. \tag{3.13}$$

Let us denote

$$\vartheta = ((0, \dots, 0), 1, \dots, 1, (0, \dots, 0))' \tag{3.14}$$

the parameter, corresponding to the $N(0, I_k)$ distribution, and assume at first that

$$\gamma = \vartheta. \tag{3.15}$$

Let us denote

$$\overline{x} = \frac{1}{n} \sum_{j=1}^{n} x_j, \qquad \boldsymbol{\Sigma} = \frac{1}{n} \sum_{j=1}^{n} (x_j - \overline{x})(x_j - \overline{x})' \tag{3.16}$$

and for positive constants M, α, β put

$$\begin{gathered} A_n^{(1)} = \{x^{(n)};\ \|\overline{x}\| < M\}, \qquad A_n^{(2)} = \{x^{(n)};\ \lambda_1(\boldsymbol{\Sigma}) > \alpha\}, \\ A_n^{(3)} = \{x^{(n)};\ \lambda_k(\boldsymbol{\Sigma}) < \beta\}, \end{gathered} \tag{3.17}$$

where $\lambda_1(\boldsymbol{\Sigma}) \geq \ldots \geq \lambda_k(\boldsymbol{\Sigma})$ are the characteristic roots of $\boldsymbol{\Sigma}$. Since

$$P_\vartheta[\, A_n^{(1)}\,] = P[\, \chi_k^2 > nM^2\,],$$

from (3.7) and (3.10) we obtain that

$$\limsup_{n \to \infty} \frac{1}{n} \log P_\vartheta[\, A_n^{(j)}\,] < -\eta \tag{3.18}$$

for $j = 1$, if M is sufficiently large. Further,

$$P_\vartheta[\, A_n^{(2)}\,] \;\leq\; P[\, \mathrm{tr}(\boldsymbol{\Sigma}) > \alpha\,] \;=\; P[\, \chi^2_{k(n-1)} > n\alpha\,]$$

and from (3.7), (3.11) we get that (3.18) holds for $j = 2$, if α is sufficiently large. Since according to Theorem 3.3.8 in [13] the random variable $|n\boldsymbol{\Sigma}|$ is distributed as $\prod_{j=1}^{k} \chi^2_{n-j}$,

$$P_\vartheta[\, A_n^{(3)}\,] \;\leq\; P_\vartheta[\, A_n^{(2)}\,] \;+\; P_\vartheta[\, |\boldsymbol{\Sigma}| < \alpha^{k-1}\beta\,] \;\leq\; P_\vartheta[\, A_n^{(2)}\,] \;+\; kP\Big[\chi^2_{n-k} < n\Big(\alpha^{k-1}\beta\Big)^{1/k}\Big].$$

Hence taking into account Lemma 2.1, (3.7) and (3.12) we see that (3.18) holds for $j = 3$, if $\beta > 0$ is sufficiently small. Thus with the notation

$$A_n = \bigcup_{j=1}^{3} A_n^{(j)} \tag{3.19}$$

according to (3.18) and Lemma 2.1 the inequality (1.19) is true.

Now we shall utilize the fact that the likelihood function

$$\begin{aligned} \frac{1}{n} \log L(x^{(n)}, \gamma^*) = -\frac{k}{2} \log 2\pi - \frac{1}{2} \sum_{j=1}^{k} \log \lambda_j(\gamma^*) - \\ -\frac{1}{2} \mathrm{tr}\Big(\mathbf{V}(\gamma^*)^{-1} [\boldsymbol{\Sigma} + (\overline{x} - \mu^*)(\overline{x} - \mu^*)']\Big), \end{aligned}$$

where $\lambda_1(\gamma^*) \geq \ldots \geq \lambda_1(\gamma^*)$ are the characteristic roots of $\mathbf{V}(\gamma^*)$. Thus

$$\frac{1}{n}\log L(x^{(n)}, \vartheta) = -\frac{k}{2}\log 2\pi - \frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}) - \frac{1}{2}\|\overline{x}\|^2 ,$$

which together with (3.17) and (3.19) implies (1.20) with $N = k+1$. Further, for every symmetric positive definite $k \times k$ matrices $\mathbf{V}(\gamma^*)$, $\boldsymbol{\Sigma}$ according to Theorem 1.10.2 in [13], p. 22

$$\mathrm{tr}[\mathbf{V}(\gamma^*)^{-1}\boldsymbol{\Sigma}] \geq \sum_{j=1}^{k} \frac{\lambda_j(\boldsymbol{\Sigma})}{\lambda_j(\gamma^*)} \tag{3.20}$$

and for $x^{(n)} \in X^n - A_n$ therefore

$$\frac{1}{n}\log L(x^{(n)}, \gamma^*) \leq -\frac{k}{2}\log(2\pi) - \frac{1}{2}\sum_{j=1}^{k} g\big(\lambda_j(\gamma^*)\big) - \frac{1}{2}(\overline{x}-\mu^*)'\,\mathbf{V}(\gamma^*)^{-1}(\overline{x}-\mu^*) , \tag{3.21}$$

where

$$g(z) = \log z + \beta/z . \tag{3.22}$$

Obviously

$$g(z) \geq g(\beta) , \tag{3.23}$$

$$g(z) \ \to +\infty \ if \ \ z \to 0^+ \ \ or \ \ z \to +\infty . \tag{3.24}$$

Hence if we denote

$$\Gamma_1 = \{\gamma^* \in \Xi;\ \tilde{\beta} \leq \lambda_k(\gamma^*) ,\ \lambda_1(\gamma^*) \leq \tilde{\alpha}\} ,$$

then combining (3.21) – (3.24) we see that if c is a real number, then there exist positive real numbers $\tilde{\beta} < \tilde{\alpha}$ such that

$$\frac{1}{n}\log L(x^{(n)}, \Xi - \Gamma_1) < c \tag{3.25}$$

whenever $x^{(n)} \in X^{(n)} - A_n$. Let

$$\Gamma_2 = \{\, \gamma^* \in \Xi;\ \|\mu^*\| \leq \tilde{M}\} .$$

Utilizing for $\theta^* \in \Gamma_1 - \Gamma_2$ and $x^{(n)} \in X^{(n)} - A_n$ the inequalities

$$(\overline{x}-\mu^*)'\,\mathbf{V}(\gamma^*)^{-1}(\overline{x}-\mu^*) \geq \lambda_1(\gamma^*)^{-1}\|\overline{x}-\mu^*\|^2 ,$$
$$\|\overline{x}-\mu^*\| \geq \|\mu^*\| - \|\overline{x}\| > \tilde{M} - M$$

and (3.23), we obtain from (3.21) that for $\tilde{M} > 0$ sufficiently large

$$\frac{1}{n}\log L(x^{(n)}, \Gamma_1 - \Gamma_2) < c \tag{3.26}$$

whenever $x^{(n)} \in X^{(n)} - A_n$. Since the set $\Gamma_c = \Gamma_1 \bigcap \Gamma_2$ is compact and according to (3.25), (3.26) the inequality (1.21) holds, (A III) is proved provided that (3.15) holds.

Let $\gamma \neq \vartheta$. If we denote for $\gamma^* \in \Xi$ by $h(\gamma^*)$ the unique parameter from Ξ, corresponding to $N[\mathbf{V}(\gamma)^{-1/2}(\mu^*-\mu), \mathbf{V}(\gamma)^{-1/2}\mathbf{V}(\gamma^*)\mathbf{V}(\gamma)^{-1/2}]$ and put $z_j = T(x_j) = \mathbf{V}(\gamma)^{-1/2}(x_j - \mu)$, then

$$\frac{1}{n}\log L(x^{(n)},\gamma^*) = \frac{1}{n}\log L(z^{(n)},h(\gamma^*)) - \frac{1}{2}\log|\mathbf{V}(\gamma)| \tag{3.27}$$

$$P_\vartheta[\,A_n\,] = P_\gamma[\,B_n\,]\,, \qquad B_n = \{\,(x_1,\dots,x_n);\ (T(x_1),\dots,T(x_n)) \in A_n\}\,.$$

Since the set $h^{-1}(\Gamma)$ is compact if Γ is compact, validity of (A III) follows from its validity in the case (3.15).

(A IV) One can choose a number $\delta_0 > 0$ such that $V = \{\gamma^* \in R^m;\ \|\gamma^* - \gamma\| < \delta_0\}$ is a subset of Ξ, and

$$\lambda_1(\gamma^*) < 2\lambda_1(\gamma)\,, \qquad |\mathbf{V}(\gamma^*)|^{-1/2} < 2|\mathbf{V}(\gamma)|^{-1/2} \tag{3.28}$$

whenever $\gamma^* \in V$. If $\gamma^* \in V$, then

$$(x-\mu^*)'\,\mathbf{V}(\gamma^*)^{-1}(x-\mu^*) \ge \frac{\|x-\mu^*\|^2}{2\lambda_1(\gamma)}\,. \tag{3.29}$$

Denoting $B = \{x \in R^k : \|x-\mu\| > 1 + 2\delta_0\}$ we see that for $x \in B$, $\gamma^* \in V$

$$\begin{aligned} \|x-\mu^*\|^2 &\ge \Big[\,\|x-\mu\| - \|\mu-\mu^*\|\,\Big]^2 \ge \\ &\ge \|x-\mu\|^2 - 2\delta_0\|x-\mu\| > \|x-\mu\|\,. \end{aligned} \tag{3.30}$$

Since (AR II) and the separability of R^m imply the measurability of $L(x,V)$ and this function is bounded on $R^k - B$, combining (3.28) – (3.30) we easily obtain validity of (A IV).

(A V) According to (3.27) we assume, that (3.15) holds. Since (3.16) is the MLE of mean and covariance matrix,

$$\begin{aligned} \frac{1}{n}\log\frac{L(x^{(n)},\Xi)}{L(x^{(n)},\vartheta)} &= -\frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{k}{2} + \frac{1}{2}\mathrm{tr}(\boldsymbol{\Sigma}) + \frac{1}{2}\|\overline{x}\|^2 < \\ &< -\frac{k}{2}\log\lambda_k(\boldsymbol{\Sigma}) + \frac{k}{2}\lambda_1(\boldsymbol{\Sigma}) + \frac{1}{2}\|\overline{x}\|^2 \end{aligned}$$

and (1.24) follows from (3.18) and Lemma 2.1.

(AR VI) If $\gamma = (\mu', \sigma_1, \dots, \sigma_k, \rho')'$, $\gamma_n = (\mu_n{}', \sigma_1^{(n)}, \dots, \sigma_k^{(n)}, \rho_n{}')'$ belong to Ξ, then (3.20) implies that

$$K(\gamma,\gamma_n) \ge \frac{1}{2}(\mu-\mu_n)'\,\mathbf{V}(\gamma_n)^{-1}(\mu-\mu_n) - \frac{k}{2} + \frac{1}{2}\sum_{j=1}^{k} g\Big(\frac{\lambda_j(\gamma_n)}{\lambda_j(\gamma)}\Big)\,, \tag{3.31}$$

where the function g is determined by (3.22) with $\beta = 1$. Let us assume that (1.25) holds and

$$\lim_{n\to\infty}\mu_n = \tilde{\mu}\,, \qquad \lim_{n\to\infty}\mathbf{V}(\gamma_n) = \tilde{\mathbf{V}}\,, \qquad \mathbf{R}(\rho) = \tilde{\mathbf{R}}\,.$$

If $|\tilde{v}_{ij}| = +\infty$ for some i, j, then $\lambda_1(\gamma_n) \ge \max_r(\sigma_r^{(n)})^2 \to +\infty$ and from (3.31), (3.23) and (3.24) we obtain that

$$\lim_{n\to\infty} K(\gamma,\gamma_n) = +\infty \tag{3.32}$$

which is a contradiction with (1.25). Proceeding similarly in the case $|\tilde{\mathbf{V}}| = 0$ or $|\tilde{\mu}_i| = +\infty$ for some i, we obtain (3.32). Hence $\lim_{n\to\infty} \gamma_n = \tilde{\gamma} \in \Xi$, from which we easily get (1.26) and (AR VI) is proved.

(A VII) Since according to (3.27) the distribution of $2\log(\, L(x^{(n)}, \Xi)/L(x^{(n)}, \gamma)\,)$ does not depend on γ, the assumption (A VII) is fulfilled. □

Example 1. Testing independence of subvectors.

In accordance with (1.2) assume that $\Theta = \Xi$ is determined by (3.2). Let $r > 1$ and

$$\{1, \ldots, k\} = \bigcup_{j=1}^{r} \{i_j, \ldots, i_{j+1} - 1\}\,, \qquad \mathbf{x}_j = (x_{i_j}, x_{i_j+1}, \ldots, x_{i_{j+1}-1})'\,.$$

If $\gamma \in \Theta$ and P_γ is the true distribution of $\mathbf{x}$, then the covariance matrix of the vector $\mathbf{x}$

$$\mathbf{V}(\gamma) = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} & \ldots & \mathbf{\Sigma}_{1r} \\ \ldots & & \vdots & \\ \mathbf{\Sigma}_{r1} & \mathbf{\Sigma}_{r2} & \ldots & \mathbf{\Sigma}_{rr} \end{pmatrix},$$

where $\mathbf{\Sigma}_{ij} = \operatorname{cov}(\mathbf{x}_i, \mathbf{x}_j)$. If

$$H = \{\, \gamma \in \Theta;\ \mathbf{\Sigma}_{ij} = \mathbf{0} \textit{ for all } i \neq j \}\,, \tag{3.33}$$

i.e., H is the hypothesis that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are independent, then according to [9], p. 413

$$T_n = 2\log \frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, H)} = n \log \Big[\Big(\prod_{j=1}^{r} |\mathbf{S}_{jj}| \Big) / |\mathbf{S}| \Big] \tag{3.34}$$

where $\mathbf{S}_{ij}$ is the submatrix of $\mathbf{S} = n\mathbf{\Sigma}$, corresponding to the vectors $\mathbf{x}_i$, $\mathbf{x}_j$. If $n \to \infty$, then according to Theorem 3.1.(II) the statistic (3.34) is optimal in the sense of exact slopes. As shown in [9], p. 414, under (3.33) the null distribution of (3.34) tends to χ^2_d with $d = 2^{-1}(k^2 - \sum_{j=1}^{r} (i_{j+1} - i_j)^2)$ degrees of freedom, which according to Theorem 3.1(II) means that the approximate slope of (3.34) exists and equals its exact slope.

Example 2. Testing equality of covariance matrices.

Let Ξ be the set (3.2), $q > 1$ and $\Theta = \Xi^q$. Hypothesis of equality of covariance matrices can be written as (cf. (3.3))

$$H = \{\, \theta = (\theta_1, \ldots, \theta_q) \in \Theta;\ \mathbf{V}(\theta_1) = \mathbf{V}(\theta_2) = \ldots = \mathbf{V}(\theta_q)\,\}\,. \tag{3.35}$$

As shown in [9], pp. 403 – 404, in this case

$$T_u(x^{(u)}) = 2\log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, H)} = \log \lambda^*\,, \qquad \lambda^* = \left| \frac{1}{N}\mathbf{A} \right|^N \Big/ \prod_{j=1}^{q} |\mathbf{\Sigma}_j|^{n_j} \tag{3.36}$$

where for the sake of brevity we use the notation $N = n_u$, $n_j = n_u^{(j)}$, and where $\mathbf{\Sigma}_j = (1/n_j)\mathbf{S}_j$ is the sample covariance matrix of the sample from the jth population, and $\mathbf{A} = \mathbf{S}_1 + \ldots + \mathbf{S}_q$. As pointed out in [13], pp. 225 and 317, to achieve the unbiasedness, instead of T_u the modified statistic

$$(3.37) \qquad \tilde{T}_u = \log\lambda\,, \qquad \lambda = \left|\frac{1}{N-q}\mathbf{A}\right|^{N-q} \Big/ \prod_{j=1}^{q}\left|\frac{1}{n_j-1}\mathbf{S}_j\right|^{n_j-1}$$

is used for testing (3.35). Let us assume that (A I) holds. If $\theta \in \Theta - H$, then according to Theorem 3.1 with probability 1

$$\lim_{u\to\infty}\frac{1}{n_u}\tilde{T}_u = \lim_{u\to\infty}\frac{1}{n_u}T_u = 2J(\theta)\,.$$

Since for u sufficiently large $\tilde{T}_u \le T_u + c$, the optimality of $\tilde{T}_u$ in the sense of exact slopes can be easily proved by means of (1.13) and Lemma 2.3.

Now let

$$(3.38) \qquad \Xi = \{(\mu,\sigma);\ \mu \in R\,,\ \sigma > 0\}\,,$$

and for $\gamma = (\mu,\sigma) \in \Xi$ let

$$(3.39) \qquad f(,\gamma) = (2\pi)^{-1/2}\sigma^{-1}\exp[-(2\sigma^2)^{-1}(x-\mu)^2]$$

be density of the one-dimensional normal distribution $\overline{P}_\gamma = N(\mu,\sigma^2)$ with the mean μ and the variance σ^2.

Theorem 3.2. *With the notation (3.38), (3.39) and (3.6) the assumptions (AR II), (A III) – (A V), (AR VI), (A VII) are fulfilled, and the assertion (II) of Theorem 3.1 is true.*

The proof can be performed similarly as in Theorem 3.1 and is left to the reader.

Before presenting the next example, we recall that if $\ll$ is a partial order on a set S, then a function $\mu(.)$ is said to be isotone with respect to $\ll$, if $\mu(x) \le \mu(y)$ whenever $x \ll y$.

Example 3. Testing isotonicity of means.

Let Ξ be the set (3.38), $\Theta = \Xi^q$, $q > 1$, $\ll$ be a partial order on $S = \{1,\dots,q\}$ and

$$H_1 = \{(\mu_1,\sigma,\dots,\mu_q,\sigma) \in \Theta;\ \textit{vector } (\mu_1,\dots,\mu_q) \textit{ is isotone and } \sigma > 0\}$$

be the hypothesis that the means of the underlying normal populations are isotone (and the usual assumption of equality of variances is imposed). Let

$$H_2 = \{(\mu_1,\sigma,\dots,\mu_q,\sigma) \in \Theta;\ \sigma > 0\}$$

be the hypothesis, which places no restrictions on means, but still assumes the equality of variances. It is easy to see that in this case

$$(3.40) \qquad T_u(x^{(u)}) = 2\log\frac{L(x^{(u)},H_2)}{L(x^{(u)},H_1)} = -2\log\lambda_{12}\,,$$

where

$$\lambda_{12} = \left(\frac{\hat{\sigma}_2}{\hat{\sigma}_1}\right)^{n_u}, \qquad \hat{\sigma}_2^2 = \frac{1}{n_u}\sum_{j=1}^{q}\sum_{i=1}^{n_u^{(j)}}(x_i^{(j)} - \overline{x}_j)^2$$

$$\hat{\sigma}_1^2 = \frac{1}{n_u} \sum_{j=1}^{q} \sum_{i=1}^{n_u^{(j)}} (x_i^{(j)} - \hat{\mu}_j^*)^2$$

and $\hat{\mu}^*$ is the unique vector from $C = \{\mu \in R^q;\ \mu\ is\ isotonic\}$, for which

$$\sum_{j=1}^{q} n_u^{(j)} (\overline{x}_j - \hat{\mu}_j^*)^2 = \inf\{ \sum_{j=1}^{q} n_u^{(j)} (\overline{x}_j - \mu_j)^2;\ \mu \in C \}\,.$$

If (A I) holds, then according to Theorem 3.2 the statistic (3.40) is optimal in the sense of exact slopes for testing H_1 against H_2. We remark that the null distribution of $S_{12} = 1 - \lambda_{12}^{2/n_u}$ can be found in Theorem 2.7 of [11].

Example 4. Testing the hypothesis $\mu \leq \mu_0$ by means of Student's t-statistic.

Let us assume that $\Theta = \Xi$ is the set (3.40), μ_0 is a chosen real number and

$$H = \{(\mu, \sigma) \in \Theta;\ \mu \leq \mu_0 \}\,. \tag{3.41}$$

It is easy to see that the likelihood ratio test statistic

$$T_n(x_1, \ldots, x_n) = 2 \log \frac{L(x_1, \ldots, x_n, \Theta)}{L(x_1, \ldots, x_n, H)} = \begin{cases} 0 & \overline{x} \leq \mu_0\,, \\ n \log \left[1 + \dfrac{\tilde{T}_n^2}{n-1} \right] & \overline{x} > \mu_0\,, \end{cases}$$

where with the notation $s^2 = (1/n) \sum_{j=1}^{n} (x_j - \overline{x})^2$

$$\tilde{T}_n = \tilde{T}_n(x_1, \ldots, x_n) = \frac{\overline{x} - \mu_0}{s} \sqrt{(n-1)}$$

is the usual Student's t-statistic, for which (1.6) is distribution function of the Student distribution with n-1 degrees of freedom. Thus if $\tilde{T}_n(s) > 0$, then the level attained by $\tilde{T}_n$ is the same, as the level attained by T_n. This together with the otimality of T_n (following from Theorem 3.2) means that the Student t-statistics is optimal in the sense of exact slopes for testing the hypothesis (3.41).

4. APPLICATION TO THE EXPONENTIAL DISTRIBUTION

Let

$$\Xi = \{(\mu, \sigma);\ \mu \in R\,,\ \sigma > 0 \} \tag{4.1}$$

and for $\gamma = (\mu, \sigma) \in \Xi$ let

$$\overline{P}_\gamma = E(\mu, \sigma) \tag{4.2}$$

be the exponential distribution, defined by means of the density

$$f(x, \sigma) = \begin{cases} \sigma^{-1} \exp[-(x - \mu)/\sigma] & x \geq \mu\,, \\ 0 & x < \mu \end{cases} \tag{4.3}$$

with respect to the dominating Lebesgue measure ν on the real line.

Theorem 4.1. *Let $q \geq 1$ be an integer, (A I) holds and $\Theta = \Xi^q$.*

(I) The assumptions (A II) – (A VII) are fulfilled with the notation (4.1), (3.6), (4.2), (4.3).

(II) Let us assume that (1.4) hold, $\Omega_i = \Theta \cap C_i$, C_1 is a closed subset of R^{2q} and C_2 is either closed or open. Let $\theta \in \Omega_1 - \Omega_0$. If T_u is the statistic (1.28), then (1.29) holds a.e. P_θ, T_u is optimal in the sense of exact slopes (i.e., $C(\theta) = 2J(\theta)$) and the assertions (I), (II) of Theorem 1.3 are true.

Proof. It is obvious from Theorem 1.1 that it is sufficient to prove the assertion (I).

(A II) These conditions follow from Lemma 2.4.

(A III) Let us denote

$$\hat{\mu} = \min\{x_1, \dots, x_n\}, \qquad \overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j\,, \qquad \hat{\sigma} = \overline{x} - \hat{\mu} \tag{4.4}$$

and for $x^{(n)} = (x_1, \dots, x_n) \in R^n$ and positive constatnts M, $\alpha < \beta$ put (cf. (3.13))

$$\begin{aligned} A_n^{(1)} &= \{x^{(n)};\ \hat{\mu} < \mu\}\,, & A_n^{(2)} &= \{x^{(n)};\ \hat{\mu} > \mu + M\}\,, \\ A_n^{(3)} &= \{x^{(n)};\ \hat{\sigma} < \alpha\}\,, & A_n^{(4)} &= \{x^{(n)};\ \hat{\sigma} > \beta\}\,. \end{aligned} \tag{4.5}$$

It is obvious from (4.3) that

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\gamma(A_n^{(j)}) < -\eta \tag{4.6}$$

for $j = 1$. Since $P_\gamma[\,\hat{\mu} > \mu + M] = (1 - F(M/\sigma))^n$, where F is the distribution function of $E(0,1)$, choosing M sufficiently large we see that (4.6) holds for $j = 2$. As it is shown in [5], pp. 166 – 167 (cf. also [3] and [4]), the random variables $Y_j = 2(n+1-j)(X_n^{(j)} - X_n^{(j-1)})/\sigma$, $j = 1, \dots, n$, where $X_n^{(j)}$ is the jth order statistic and $X_n^{(0)} = \mu$, are independent, and each of them is chi-square distributed with 2 degrees of freedom. This means that

$$\hat{\sigma} = \frac{1}{n}\sum_{j=1}^{n}(x_j - \mu) - (X_n^{(1)} - \mu) = \frac{\sigma}{2n}\sum_{j=2}^{n} Y_j = \frac{\sigma}{2n}\chi^2_{2(n-1)}\,, \tag{4.7}$$

which together with (3.7) and Lemma 3.1 yields the validity of (4.6) for $j = 3, 4$, if $\alpha > 0$ is sufficiently small and $\beta > 0$ is sufficiently large. Hence making use of Lemma 2.1 we see that for this choice of the constants with the notation

$$A_n = \bigcup_{j=1}^{4} A_n^{(j)} \tag{4.8}$$

the inequality (1.19) holds. Since

$$\frac{1}{n}\log L(x^{(n)}, \gamma^*) = \begin{cases} -\left[\log\sigma^* + \dfrac{\overline{x} - \mu^*}{\sigma^*}\right] & \hat{\mu} \geq \mu^*\,, \\ -\infty & \hat{\mu} < \mu^*\,, \end{cases} \tag{4.9}$$

the validity of (1.20) follows from (4.8), (4.5) and (4.4).

Further, denoting

$$\Gamma_1 = \{\gamma^* = (\mu^*, \sigma^*) \in \Xi;\ \mu^* \in \langle \mu - \mathrm{e}^{-c}, \mu + M\rangle\}$$

and taking into account (4.9), (4.8), (4.5), (4.4), and (3.23) we see that for $x^{(n)} \in R^n - A_n$ (3.25) holds. If $\mu^* \le \hat{\mu}$, then $\overline{x} - \mu^* \ge \hat{\sigma}$ and for $x^{(n)} \in R^n - A_n$

$$\frac{1}{n} \log L(x^{(n)}, \gamma^*) \le - \left[\log \sigma^* + \frac{\alpha}{\sigma^*} \right] .$$

Hence taking into account (3.24) we see that there exist positive numbers $\alpha_1 \in (0, \alpha)$, β_1 such that with the notation

$$\Gamma_2 = \{ \gamma^* \in \Xi ;\ \sigma^* \in \langle \alpha - \alpha_1 \, , \, \beta + \beta_1 \rangle \}$$

the inequality (3.26) holds whenever $x^{(n)} \in R^n - A_n$. Since the set $\Gamma_c = \Gamma_1 \cap \Gamma_2$ is compact and (3.25), (3.26) imply (1.21), the validity of (A III) is proved.

(A IV) If $\delta \in (0, \sigma)$ and V is the set (1.22), then $V \subset \tilde{V}$, where $\tilde{V} = \{ \gamma^* \in \Xi ;\ |\mu^* - \mu| < \delta \, ,\ |\sigma^* - \sigma| < \delta \}$. Since one can easily verify that $L(x, \tilde{V}) \le G(x, \tilde{V})$, where $\int G(x, \tilde{V}) \, \mathrm{d}x < +\infty$, the validity of (A IV) is obvious.

(A V) As pointed out in [6], p. 211, the estimate $\hat{\gamma} = (\hat{\mu}, \hat{\sigma})$ defined by (4.4) is the MLE of the unknown parameter γ, if $n > 1$. This together with (4.9) means that for $n > 1$ with probability 1

$$\frac{1}{n} \log \frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \gamma)} = - \left[\log \frac{\hat{\sigma}}{\sigma} + 1 \right] + \frac{\overline{x} - \mu}{\sigma} .$$

Thus with the notation $\vartheta = (0, 1)$

$$\mathcal{L} \left[\frac{1}{n} \log \frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \gamma)} \big| \, \overline{P}_\gamma \right] = \mathcal{L} \left[\frac{1}{n} \log \frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \vartheta)} \big| \, \overline{P}_\vartheta \right] \tag{4.10}$$

and

$$P_\vartheta \left[\frac{1}{n} \log \frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \vartheta)} \ge \varepsilon \right] \le P_\vartheta \left[\hat{\mu} \ge \frac{\varepsilon}{2} \right] + P_\vartheta \left[\Psi(\hat{\sigma}) \ge \frac{\varepsilon}{2} \right] ,$$

where $\Psi(\hat{\sigma}) = -\log \hat{\sigma} + \hat{\sigma}$. Since Ψ is decreasing on (0.1), increasing on $(1, +\infty)$ and $\Psi(z) \to +\infty$ if either $z \to 0^+$ or $z \to +\infty$, taking into account (4.6) and Lemma 2.1 one can easily prove (A V).

(A VI) Since

$$K(\gamma, \gamma^*) = \begin{cases} \log \dfrac{\sigma^*}{\sigma} + \dfrac{\mu - \mu^*}{\sigma^*} + \dfrac{\sigma - \sigma^*}{\sigma^*} & \mu \ge \mu^* \\ +\infty & \mu < \mu^* \end{cases}$$

from (1.25) we obtain that for all n sufficiently large

$$K(\gamma, \gamma_n) \ge -g(\sigma / \sigma_n) \, ,$$

where $g(x) = \log x - (x - 1)$. Since $g(x) < 0$ for $x \ne 1$ and $g(x) \to -\infty$ if either $x \to 0^+$ or $x \to +\infty$, we see that $\sigma / \sigma_n \to 1$ and (1.26) holds.

(A VII) The validity of this condition follows from validity of (4.10) for $n > 1$. □

Example 5. Testing the hypothesis $\mu \le \mu_0$ by means of an F-statistic.

Let us assume that $\Theta = \Xi$ is the set (4.1), μ_0 is a chosen number and

$$H = \{(\mu, \sigma) \in \Theta;\ \mu \leq \mu_0\}\,. \tag{4.11}$$

As stated in [5], p.312, for testing this hypothesis the statistic (cf. (4.4))

$$F_{n,k} = \frac{n(k-1)(\hat{\mu} - \mu_0)}{\sum_{j=2}^{k} (X_n^{(j)} - \hat{\mu}) + (n-k)(X_n^{(k)} - \hat{\mu})} \qquad 2 \leq k \leq n$$

can be used (and in such a case G_n in (1.6) is an F distribution function). It is easy to see that the MLE $\tilde{\theta} = (\tilde{\mu}, \tilde{\sigma})$ of the unknown parameter from H is determined by the formulas

$$\tilde{\mu} = \begin{cases} \hat{\mu} & \hat{\mu} \leq \mu_0\,, \\ \mu_0 & \hat{\mu} > \mu_0\,, \end{cases} \qquad \tilde{\sigma} = \overline{x} - \tilde{\mu}\,.$$

Hence the likelihood ratio test statistic for testing (4.11)

$$T_n(x^{(n)}) = 2 \log \frac{L(x^{(n)}, \Theta)}{L(x^{(n)}, H)} = \begin{cases} 0 & \hat{\mu} \leq \mu_0\,, \\ 2n \log \left[1 + \dfrac{\tilde{T}_n}{n-1}\right] & \hat{\mu} > \mu_0\,, \end{cases} \tag{4.12}$$

where $\tilde{T}_n = F_{n,n}$. Let us denote $\tilde{L}_n(s)$ the level attained by $\tilde{T}_n$ and $L_n(s)$ the level attained by T_n. If $T_n(s) > 0$, then (4.12) implies that $\tilde{L}_n(s) = L_n(s)$ and from Theorem 4.1 we obtain that the statistic $\tilde{T}_n = F_{n,n}$ is optimal in the sense of exact slopes for testing (4.11).

5. APPLICATION TO THE LAPLACE DISTRIBUTION

Let

$$\Xi = \{(\mu, \sigma);\ \mu \in R\,,\ \sigma > 0\} \tag{5.1}$$

and for $\gamma = (\mu, \sigma) \in \Xi$ let

$$\overline{P}_\gamma = L(\mu, \sigma) \tag{5.2}$$

be the Laplace distribution, defined by means of the density

$$f(x, \gamma) = \frac{1}{2\sigma} \exp\left[-\frac{|x - \mu|}{\sigma}\right] \tag{5.3}$$

with respect to the dominating Lebesgue measure ν on the real line.

Theorem 5.1. *Let $q \geq 1$ be an integer, (A I) holds and $\Theta = \Xi^q$.*

(I) The assumptions (AR II), (A III) – (A V), (AR VI) and (A VII) are fulfilled with the notation (5.1), (3.6), (5.2), (5.3).

(II) If (1.4) holds, T_u is the statistic (1.28) and $\theta \in \Omega_1 - \Omega_0$, then (1.29) holds a.e. P_θ with $C(\theta) = 2J(\theta)$ (i.e., T_u is optimal in the sense of exact slopes) and the assertions (I) and (II) of Theorem 1.3 are true.

In proving the theorem we shall use the following lemma.

Lemma 5.1. *Let $x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(n)}$ be ordering of numbers $x_1, \ldots, x_n$ according to their magnitude. Let us denote*

$$\hat{\mu} = \begin{cases} x^{(k+1)} & n = 2k+1\,, \\ \dfrac{x^{(k)} + x^{(k+1)}}{2} & n = 2k \end{cases} \tag{5.4}$$

the median of the numbers $x_1, \ldots, x_n$.

(I) If μ is a real number, then

$$\sum_{j=1}^{n} |x_j - \hat{\mu}| \leq \sum_{j=1}^{n} |x_j - \mu|\,. \tag{5.5}$$

(II) If $k > 1$, then

$$\sum_{j=1}^{n} |x_j - \hat{\mu}| = \begin{cases} \sum\limits_{j=1}^{k} j(x^{(j+1)} - x^{(j)}) + \sum\limits_{j=k+1}^{2k} (n-j)(x^{(j+1)} - x^{(j)}) & n = 2k+1\,, \\ \sum\limits_{j=1}^{k-1} j(x^{(j+1)} - x^{(j)}) + \sum\limits_{j=k}^{2k-1} (n-j)(x^{(j+1)} - x^{(j)}) & n = 2k\,. \end{cases} \tag{5.6}$$

The relation (5.5) is according to [7], p. 26 proved in [8], and the proof of (5.6) is left to the reader.

Proof of Theorem 5.1. (AR II) This regularity assumption is obviously true.

(A III) Let $\hat{\mu}$ be the sample median (5.4) and

$$\hat{\sigma} = \frac{1}{n} \sum_{j=1}^{n} |x_j - \hat{\mu}|\,. \tag{5.7}$$

Let us denote for positive real constants α, $\alpha < \beta$

$$A_n^{(1)} = \{x^{(n)} \in R^n;\ |\hat{\mu}| > \alpha\}\,, \quad A_n^{(2)} = \{x^{(n)} \in R^n;\ |\hat{\sigma}| < \alpha\}\,, \quad A_n^{(3)} = \{x^{(n)} \in R^n;\ |\hat{\sigma}| > \beta\}\,. \tag{5.8}$$

Since the random variables $(x_j - \mu)/\sigma$ are $L(0,1)$ distributed, in proving (4.6) we may assume that

$$\gamma = \vartheta = (0,1)\,. \tag{5.9}$$

Since the distribution function of $L(0,1)$ equals $2^{-1}\mathrm{e}^{x}$ if $x \leq 0$ and $1 - 2^{-1}\mathrm{e}^{-x}$ if $x > 0$, utilizing the formula for density of the jth order statistic we get that the moment generating function of $X_n^{(j)}$

$$\begin{aligned} \phi_j(t) &= \int_{-\infty}^{+\infty} \mathrm{e}^{tx}\, n \binom{n-1}{j-1} F(x)^{j-1} (1 - F(x))^{n-j} f(x)\, \mathrm{d}x = \\ &= n \binom{n-1}{j-1} G(t,n,j) \end{aligned} \tag{5.10}$$

where for $1 < j < n$ and $0 < t < d = d(j,n) = min\{j, n-j+1\}$

$$G(t,n,j) < \int_{-\infty}^{0} \mathrm{e}^{(t+j)x}\,\mathrm{d}x + \int_{0}^{+\infty} \mathrm{e}^{-((n-j+1)-t)x}\,\mathrm{d}x < 2/(d-t)\,.$$

Hence utilizing (5.10) and Markov's inequality we see that for $0 < a^{-1} < d(j,n)$

$$\begin{aligned} \log P\left[X_n^{(j)} > a\right] &\leq \log n + \log \binom{n-1}{j-1} + \inf\left\{\log \frac{G(t,n,j)}{\mathrm{e}^{at}};\ 0 < t < d\right\} \leq \\ &\leq \log n + \log \binom{n-1}{j-1} + \log 2 - ad + 1 + \log a\,. \end{aligned} \tag{5.11}$$

Since $\mathcal{L}(X) = \mathcal{L}(-X)$ under (5.9), obviously

$$P[\,|\hat{\mu}| > a\,] = 2P[\,\hat{\mu} > a\,]$$

and making use of (5.4), (5.11), Lemma 2.1 and the Stirling formula one can show that (4.6) holds for $j = 1$, if a is sufficiently large.

Let $z_j = |x_j|$. If $\tilde{\mu}$ denotes the median of $z_1, \ldots, z_n$ then according to (5.5)

$$\frac{1}{n}\sum_{j=1}^{n} |x_j - \hat{\mu}| \geq \frac{1}{n}\sum_{j=1}^{n} \Big||x_j| - |\hat{\mu}|\Big| \geq \frac{1}{n}\sum_{j=1}^{n} |z_j - \tilde{\mu}|\,. \tag{5.12}$$

We know that (cf. (4.2))

$$\mathcal{L}(z_j) = E(0,1) \tag{5.13}$$

and if $Z_n^{(j)}$ is the jth order statistic generated by $z_1, \ldots, z_n$, then according to [5], pp. 166-167 the random variables $Y_j = 2(n+1-j)(Z_n^{(j+1)} - Z_n^{(j)})$, $j = 1, \ldots, n-1$ are independent, each having chi-square distribution with 2 degrees of freedom. This together with (5.6) and (5.12) means that

$$P[\,|\hat{\sigma}| \leq \alpha\,] \leq P[\,\frac{1}{4}\chi^2_{n-1} \leq n\alpha\,]\,.$$

Hence making use of (3.7) and (3.12) we get that (4.6) holds for $j = 2$ if α is sufficiently small.

Since according to (5.5)

$$P[\,|\hat{\sigma}| > \beta\,] \leq P\left[\frac{1}{n}\sum_{j=1}^{n} z_j > \beta\right] \leq P[\,Z_n^{(1)} > \beta/2\,] + P\left[\frac{1}{n}\sum_{j=1}^{n} z_j - Z_n^{(1)} > \beta/2\right]\,,$$

taking into account (5.13) and validity of (4.6) in the case (4.1) – (4.5) we see that (4.6) holds for $j = 3$ if β is sufficiently large. Thus putting (for this appropriate choice of the constants)

$$A_n = \bigcup_{j=1}^{3} A_n^{(j)} \tag{5.14}$$

and making use of Lemma 2.1 we see that (1.19) holds.

Obviously for $\gamma^* = (\mu^*, \sigma^*) \in \Xi$

$$\frac{1}{n}\log L(x^{(n)}, \gamma^*) = -\log 2 - \log \sigma^* - \frac{1}{n\sigma^*}\sum_{j=1}^{n}|x_j - \mu^*| \tag{5.15}$$

which together with (5.14) and (5.8) implies (1.20). Since for $x^{(n)} \in X^n - A_n$ according to (5.15) and (5.5)

$$\frac{1}{n}\log L(x^{(n)}, \gamma^*) = -\log 2 - g(\sigma^*)$$

where g is the function (3.22) with α instead of β, (3.24) implies the existence of $\alpha_1 \in (0, \alpha)$, $\beta_1 \in (0, +\infty)$ such that with the set

$$\Gamma_1 = \{\gamma^* \in \Xi;\ \sigma^* \in \langle \alpha - \alpha_1, \beta + \beta_1 \rangle\}$$

the inequality (3.25) holds whenever $x^{(n)} \in X^n - A_n$. Combining (5.15) with

$$\frac{1}{n}\sum_{j=1}^{n}|x_j - \mu^*| \ge \frac{1}{n}\sum_{j=1}^{n}\Big||x_j| - |\mu^*|\Big| \ge |\mu^*| - \frac{1}{n}\sum_{j=1}^{n}|x_j|\,, \qquad \frac{1}{n}\sum_{j=1}^{n}|x_j| \le \hat{\sigma} + |\hat{\mu}|$$

we obtain the existence of a number $a_1 > 0$ for which with the set

$$\Gamma_2 = \{\gamma^* \in \Xi;\ |\mu^*| \le |\mu + a_1|\rangle\}$$

(3.26) holds whenever $x^{(n)} \in X^n - A_n$. Since the set $\Gamma = \Gamma_1 \cap \Gamma_2$ is compact and (3.25), (3.26) imply (1.21), (A III) is proved.

(A IV) If $\varepsilon \in (0, \sigma)$ and $V = (\mu - \varepsilon\,,\, \mu + \varepsilon) \times (\sigma - \varepsilon\,,\, \sigma + \varepsilon)$, then (1.23) obviously holds.

(A V) and (A VII) As stated in [7], p.26, the estimate $\hat{\gamma} = (\hat{\mu}, \hat{\sigma})$ defined by (5.4) and (5.7) is the MLE of the unknown parameter γ, if $n > 1$. This together with (5.15) yields that with probability 1

$$\frac{1}{n}\log\frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \gamma)} = -\log\frac{\hat{\sigma}}{\sigma} - 1 + \frac{1}{n\sigma}\sum_{j=1}^{n}|x_j - \mu|$$

which means that (4.10) is valid and (A VII) is proved.

Let us assume that (5.9) holds and $g(x) = x - \log x$. Then

$$P\left[\frac{1}{n}\log\frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \gamma)} \ge \varepsilon\right] \le P\left[g(\hat{\sigma}) > \frac{\varepsilon}{2}\right] + P\left[|\hat{\mu}| > \frac{\varepsilon}{2}\right]$$

and taking into account (5.8) and (4.6) we get the validity of (A V).

(AR VI) Since

$$K(\gamma, \gamma_n) = \log\frac{\sigma_n}{\sigma} + \frac{(\mu - \mu_n)}{\sigma_n} + \frac{\sigma}{\sigma_n}\exp\left[-\frac{|\mu - \mu_n|}{\sigma}\right] - 1\,, \tag{5.16}$$

each of the possibilities $\sigma_n \to 0$, $\sigma_n \to +\infty$, $|\mu_n| \to +\infty$ leads to the equality

$$\lim_{n\to\infty} K(\gamma, \gamma_n) = +\infty$$

which is a contradiction with (1.25). But if $\tilde{\gamma}$ is a limiting point of $\{\gamma_n\}$, then (5.16) and (1.25) imply that $K(\gamma, \tilde{\gamma}) = 0$, and since $\overline{P}_\gamma \ne \overline{P}_{\tilde{\gamma}}$ if $\gamma \ne \tilde{\gamma}$, (1.26) is proved. □

6. APPLICATION TO THE POISSON DISTRIBUTION

Let

$$\Xi = \langle 0, +\infty) \tag{6.1}$$

and for $\gamma \in \xi$ let $\overline{P}_\gamma$ be the Poisson distribution, defined by means of the density

$$f(j, \lambda) = \frac{e^{-\gamma}\gamma^j}{j!} \qquad j = 0, 1, 2, \dots \tag{6.2}$$

with respect to the counting measure ν on $X = \{0, 1, 2, \dots\}$, where $0^0 = 1$.

Theorem 6.1. *Let $q \geq 1$ be an integer, (A I) holds and $\Theta = \Xi^q$.*

(I) If we denote $\tau(\gamma, \gamma^*) = |\gamma - \gamma^*|$, then with the notation (6.1), (6.2) the assumptions (AR II), (A III) – (A V), (AR VI) are fulfilled.

(II) If (1.4) holds, T_u is the statistic (1.28) and $\theta \in \Omega_1 - \Omega_0$, then (1.29) holds a.e. P_θ, (1.9) holds a.e. P_θ with $C(\theta) = 2J(\theta)$ (i.e., T_u is optimal in the sense of exact slopes) and the assertions (I) and (II) of Theorem 1.3 are true.

Let us put

$$\begin{gathered} 0/0 = 1\,, \qquad x/0 = +\infty \ \text{ if } x > 0\,, \qquad 0 \log y = 0 \ \text{ if } y \geq 0\,, \\ x \log(+\infty) = +\infty \ \text{ if } x > 0\,, \end{gathered} \tag{6.3}$$

and for $\gamma \in \xi$, $x \in \langle 0, +\infty)$ let

$$g(x, \gamma) = \gamma - x + x \log(x/\gamma)\,. \tag{6.4}$$

Before carrying out the proof of the theorem we shall establish the validity of the following assertion.

Lemma 6.1. *Let the random variable ξ has Poisson distribution $\overline{P}_\gamma$.*

(I) If the real number $a \geq \gamma$, then

$$\log P[\xi \geq a] \leq -g(a, \gamma)\,. \tag{6.5}$$

(II) If $0 \leq a \leq \gamma$, then

$$\log P[\xi \leq a] \leq -g(a, \gamma)\,. \tag{6.6}$$

(III) Ifv $z \in \langle 0, \frac{1}{2})$, then

$$E[\,\exp(2zg(\xi, \gamma))\,] < \frac{2}{1 - 2z}\,. \tag{6.7}$$

P r o o f. (I) If $\gamma = 0$, then (6.5) holds. If $\gamma > 0$, then $z = \log(a/\gamma) \in \langle 0, +\infty)$ and making use of the Markov inequality we obtain that

$$P[\xi \geq a] \leq E[\,e^{z\xi}\,]\,e^{-za} = \exp[\gamma(e^z - 1) - za]\,.$$

(II) The proof is similar as in the case of (6.5).

(III) Let us assume that $\gamma > 0$ be an arbitrary but fixed real number and

$$P_1(\gamma) = P[\,g(\xi, \gamma) \geq \frac{1}{2}t\,,\ \xi > \gamma\,]\,, \qquad P_2(\gamma) = P[\,g(\xi, \gamma) \geq \frac{1}{2}t\,,\ \xi \leq \gamma\,]\,.$$

Since $g(x) \to +\infty$ if $x \to +\infty$, the set $A = \{j;\ \ j \textit{ is an integer, } j > \gamma,\ g(j,\gamma) \geq \frac{1}{2}t\,\}$ is non-empty. If we denote $a = \min\{j;\ j \in A\,\}$, then taking into account the fact that g increases on $(\gamma, +\infty)$ and making use of (6.5) we obtain the inequality

$$\log P_1(\gamma) = \log P[\xi \geq a] \leq -g(a,\gamma) \leq -\frac{1}{2}t\,. \tag{6.8}$$

Since g is decreasing on $\langle 0,\gamma)$ and $g(0) = \gamma$, similarly as in (6.8) the relation (6.6) implies that

$$\log P_2(\gamma) \leq -\tfrac{1}{2}t$$

and we see that

$$P[g(\xi,\gamma) \geq \frac{1}{2}t] \leq 2\mathrm{e}^{-t/2}\,. \tag{6.9}$$

As pointed out in [1], p. 294, for every non-negative measurable function g

$$\int g\,\mathrm{d}P = \int_0^{+\infty} P[g \geq t]\,\mathrm{d}t \tag{6.10}$$

in the sense that if either integral exists, so does the other and the two are equal. Combining (6.9) and (6.10) we get

$$\begin{aligned} E[\,\exp(2z\,g(\xi,\gamma))\,] &= \int_0^{+\infty} P\Big[g(\xi,\gamma) \geq \frac{1}{2z}\log t\Big]\ \mathrm{d}t \leq 1 + \int_1^{+\infty} 2t^{-1/(2z)}\,\mathrm{d}t = \\ &= 1 + 4z/(1-2z)\,. \end{aligned}$$

Since for $\gamma = 0$ the inequality (6.7) obviously holds, the lemma is proved. $\square$

Proof of Theorem 6.1 (I)

(AR II) This assumption is obviously true.

(A III) Let us denote

$$\hat\gamma = \frac{1}{n}\sum_{j=1}^{n} x_j\,. \tag{6.11}$$

If $\gamma = 0$, then for $A_n = \{x^{(n)} \in X^n;\ \hat\lambda > 0\,\}$ the relations (1.19) and (1.20) hold, and denoting $\Gamma_c = \langle 0, |c|+1\rangle$ we easily obtain (1.21).

Let γ be a positive real number.

If $\beta > 1$, then $z = 1 - \beta^{-1} \in (0,1)$ and taking into account Markov's inequality and (6.7) we see that

$$P_\gamma[\sum_{j=1}^{n} g(x_j,\gamma) > n\beta\,] \leq \mathrm{e}^{-zn\beta}E[\,\exp(zg(x,\gamma))\,]^n < \exp[\,-n\beta + n + n\log 2\beta\,]\,.$$

This together with (6.11), $\mathcal{L}(n\hat\gamma) = \overline{P}_{n\gamma}$ and (6.5) yields the existence of positive real numbers α, β such that with the notation

$$A_n^{(1)} = \{x^{(n)} \in X^n;\ \hat\gamma > \alpha\}\,, \qquad A_n^{(2)} = \{x^{(n)} \in X^n;\ \sum_{j=1}^{n} g(x_j,\gamma) > n\beta\} \tag{6.12}$$

the inequality (4.6) holds for $j = 1,2$. Hence if we put

$$A_n = A_n^{(1)} \cup A_n^{(2)} \tag{6.13}$$

then (1.19) follows from Lemma 2.1.

According to the Stirling formula from Section 1e.7 in [10] for $x = 1, 2, \ldots$

$$x! = \sqrt{2\pi x}(x^x)\mathrm{e}^{-x}\mathrm{e}^{\Delta(x)}\,, \qquad (x + \frac{1}{2})^{-1} < 12\Delta(x) < x^{-1}\,. \tag{6.14}$$

Since $\log(x) < x$,

$$\log x! < \tfrac{1}{2}\log 2\pi + \tfrac{1}{2}\log(x+1) + x\log x - x + 1 < \pi + 2 + x\log x\,.$$

Thus

$$\begin{aligned}\frac{1}{n}\log L(x_1, \ldots, x_n, \gamma) \quad &> \quad -\gamma + \hat{\gamma}\log\gamma - (\pi + 2) - \frac{1}{n}\sum_{j=1}^{n} x_j \log x_j = \\ &= \quad -(\pi+2) - \frac{1}{n}\sum_{j=1}^{n} g(x_j, \gamma) - \hat{\gamma}\end{aligned}$$

and from (6.12), (6.13) we obtain (1.20). Further, from (6.14) one easily finds out that

$$\log x! \geq x \log x - x$$

for all $x = 0, 1, 2, \ldots$. Hence if $\gamma^* > 0$, then

$$\begin{aligned}\log L(x_1, \ldots, x_n, \gamma^*) \quad &\leq \quad -n\gamma^* + \sum_{j=1}^{n} x_j \log\gamma^* - \sum_{j=1}^{n} x_j \log x_j + \sum_{j=1}^{n} x_j = \\ &= \quad n(\gamma - \gamma^*) + \sum_{j=1}^{n} x_j \log\frac{\gamma^*}{\gamma} - \sum_{j=1}^{n} g(x_j, \gamma)\,.\end{aligned} \tag{6.15}$$

But $g(x, \gamma) \geq 0$, and for $\gamma^* \geq \gamma$, $(x_1, \ldots, x_n) \in X^n - A_n$ therefore

$$\frac{1}{n}\log L(x_1, \ldots, x_n, \gamma^*) \leq \gamma - \gamma^* + \alpha\log\frac{\gamma^*}{\gamma}$$

which implies the statement (3) in (A III).

(A IV) Since for (6.2) the function $h(\gamma) = \log f(j, \gamma)$ is increasing on $\langle 0, j\rangle$ and decreasing on $\langle j, +\infty\rangle$, for $\delta > 0$

$$L(j, V) = \sup\{f(j,\gamma);\ 0 \leq \gamma < \delta\} = \begin{cases} \frac{\mathrm{e}^{-j}j^j}{j!} & j \leq \delta \\ \frac{\mathrm{e}^{-\delta}\delta^j}{j!} & j > \delta \end{cases} \tag{6.16}$$

and obviously $\int L(j, V)\,\mathrm{d}\nu(j) = \sum_{j=0}^{\infty} L(j, V) < +\infty$.

(A V) If $\gamma = 0$, then $L(x^{(n)}, \Xi) = L(x^{(n)}, \gamma) = 1$ with probability 1 and (A V) holds.

Let γ be a positive real number. Since the function $g(., \gamma)$ is increasing on $(\gamma, +\infty)$ and $g(x, \gamma) \to +\infty$ if $x \to +\infty$, there exists $a > \gamma$ such that $g(a, \gamma) > \max\{\gamma, \eta\}$. But $g(0, \gamma) = \gamma$ and g is decreasing on $(0, \gamma)$, which together with (6.5) implies that

$$\begin{aligned}&P_\gamma\left[\frac{1}{n}\log\frac{L(x^{(n)}, \Xi)}{L(x^{(n)}, \gamma)} \geq g(a, \gamma)\right] = P_\gamma\Big[g(\hat{\gamma}, \gamma) \geq g(a, \gamma)\Big] = P_\gamma\Big[\hat{\gamma} \geq a\Big] = \\ &= \overline{P}_{n\gamma}\Big[\xi \geq na\Big] \leq \exp[-g(na, n\gamma)] = \exp[-ng(a, \gamma)] < \exp[-n\eta]\,.\end{aligned}$$

(AR VI) It is easy to see that

$$K(\gamma,\gamma^*) = \begin{cases} \gamma^* - \gamma + \gamma \log \frac{\gamma}{\gamma^*} & \gamma^* > 0\,, \\ +\infty & \gamma^* = 0\,,\ \gamma > 0 \end{cases} \tag{6.17}$$

and (AR VI) is true, if $\gamma = 0$. Let $\gamma > 0$. Since $K(\gamma, .)$ decreases on $(0,\gamma)$, increases on $(\gamma, +\infty)$ and $K(\gamma,\gamma^*) \to +\infty$ if $\gamma^* \to 0^+$ or $\gamma^* \to +\infty$, (1.25) implies (1.26) and (AR VI) obviously holds.

(II) Validity of (1.29) a.e. P_θ follows from Theorem 1.2(I).

If $\theta \in \overline{\Omega}_0$, then the equality $C(\theta) = 2J(\theta)$ follows from (AR VI), (2.29), (2.30) and (1.13). Let $\theta \in \Omega_1 - \overline{\Omega}_0$. This according to (AR VI) means that

$$J(\theta) > 0\,. \tag{6.18}$$

Since

$$L_u(t) \le \tilde{L}_u(t)\,, \qquad \tilde{L}_u(t) = \sup\left\{ P_\theta\left[2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\theta)} \ge t \right]; \theta \in \Theta \right\}, \tag{6.19}$$

from (6.18), (I) and (1.13) we obtain that the validity of $C(\theta) = 2J(\theta)$ (and by this also the validity of the whole Theorem 6.1) will be established by proving the following lemma.

Lemma 6.2. *If q is positive integer, then with the notation (1.2), (1.3), (1.7), (6.1), (6.2) and (6.19) irrespective of the validity of (A I),*

$$\sup\{\tilde{L}_u(t); u \ge 1\} \le \exp[\,-\frac{1}{2}t(1+o(1))\,] \tag{6.20}$$

where $\lim_{t\to+\infty} o(1) = 0$.

P r o o f. Let $\theta \in \Theta$. The random variables

$$\hat{\gamma}_j = \frac{1}{n_u^{(j)}} \sum_{i=1}^{n_u^{(j)}} x_i^{(j)}\,, \qquad j = 1, \dots, q$$

are independent and $\xi_j = n_u^{(j)}\hat{\gamma}_j$ has the Poisson distribution with the parameter $n_u^{(j)}\theta_j$. If $t > 2q$, then the number

$$z = \frac{1}{2} - (q/t) \in (0, \frac{1}{2})$$

which together with Markov's inequality and (6.7) yields

$$\begin{aligned} P_\theta\left[2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\theta)} \ge t \right] &= P_\theta\Big[2\sum_{j=1}^{q} g(\xi_j, n_u^{(j)}\theta_j) \ge t \Big] \le \\ &\le \exp(-zt) E\Big[\exp(\sum_{j=1}^{q} z g(\xi_j, n_u^{(j)}\theta_j)) \Big] < \exp[\varphi(t)]\,, \end{aligned} \tag{6.21}$$

where

$$\varphi(t) = -zt + q\log\frac{2}{1-2z} = -\frac{1}{2}t[1+o(1)]\,. \tag{6.22}$$

□

We conclude the paper with discussing the hypothesis of isotonicity of means, which is under the assumptions of normality analysed in Example 3.

Example 6. Testing isotonicity of means.

Let Ξ be the set (6.1), $\Theta = \Xi^q$, $q > 1$, $\ll$ is a partial order on $S = \{1, 2, \ldots, q\}$ and

$$H_1 = \{\theta = (\theta_1, \ldots, \theta_q) \in \Theta;\ \textit{vector } \theta \textit{ is isotone and has positive coordinates}\}$$

is the hypothesis that the underlying Poisson populations are isostone. If

$$H_2 = \{\theta \in \Theta;\ \textit{vector } \theta \textit{ has positive coordinates}\}$$

is the hypothesis placing no restriction on order of the means, then according to Theorem 6.1 the statistic

$$T_{12} = 2 \log \frac{L(x^{(u)}, H_2)}{L(x^{(u)}, H_1)} \tag{6.23}$$

is optimal for testing H_1 against H_2 in the sense of exact slopes, provided that (A 1) holds (we remark that T_{12} can be computed by means of the estimates, described in [11], p.498). If $n_u^{(1)} = n_u^{(2)} = \ldots = n_u^{(q)}$ tend to $+\infty$ for $u \to \infty$, then according to Corrolary 4.2 in [11] the function (1.31) has the form (1.34) and from Theorem 6.1 we obtain that the approximate slope of (6.23) exists and equals its exact slope.

References

[1] R. R. Bahadur: Stochastic comparison of tests. *Ann. Math. Statist.*, 31(1960), 276–295.

[2] R. R. Bahadur: *Some Limit Theorems in Statistics.* SIAM, Philadelpia 1971.

[3] B. Epstein and M. Sobel: Some theorems relevant to life testing from an exponential distribution. *Ann. Math. Statist.*, 25(1954), 373–381.

[4] B. Epstein and C. K. Tsao: Some tests based on ordered observations from two exponential populations. *Ann. Math. Statist.*, 24(1953), 458–465.

[5] J. Hátle and J. Likeš: *Základy počtu pravděpodobnosti a matematické statistiky (Foundations of Probability Calculus and Mathematical Statistics).* SNTL, Praha 1974.

[6] N. L. Johnson and S. Kotz: *Continuous Univariate Distributions, Volume 1.* J. Wiley and Sons, New York 1970.

[7] N. L. Johnson and S. Kotz: *Continuous Univariate Distributions, Volume 2.* J. Wiley and Sons, New York 1970.

[8] J. M. Keynes: The principal averages and the laws of error which lead to them. *J. Roy. Statist. Soc.* 74(1911), 322–328.

[9] A. M. Kshirsagar: *Multivariate Analysis.* Marcel Dekker Inc., New York 1978.

[10] C. R. Rao: *Linear Statistical Inference and its Applications.* J. Wiley and Sons, New York 1973.

[11] T. Robertson and E. J. Wegman: Likelihood ratio tests for order restrictions in exponential families. *Ann. Statist.*, 6(1978), 485–505.

[12] F. Rublík: On Optimality of the LR tests in the sense of exact slopes. Part I: General case. *Kybernetika*, 25(1989), 13–25.

[13] M. S. Srivastava and C. G. Khatri: *An Introduction to Multivariate Statistics.* North Holland, New York 1979.