KYBERNETIKA - VOLUME 25 (1989), PAGES 13 – 25

# ON OPTIMALITY OF THE LR TESTS IN THE SENSE OF EXACT SLOPES

## Part I. General Case

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Optimality of the likelihood ratio test statistic in the sense of exact slopes is established under regularity conditions in the case, when sampling is made from $q$ populations, and the ratio $n_u^{(i)} / \sum_j n_u^{(j)}$ tends for $u \to \infty$ to a non-zero limit. The conditions are in the second part of the paper verified for particular families of distributions.

## 1. INTRODUCTION AND THE MAIN RESULTS

Let $\{\overline{P}_\gamma;\ \gamma \in \Xi\}$ be a family of probability mesures, defined on the sample space $(X, \mathcal{F})$. If we denote

(1.1) $$S = X^\infty \times \ldots \times X^\infty\,, \quad \mathcal{S} = \mathcal{F}^\infty \times \ldots \times \mathcal{F}^\infty$$

the $q$-fold product of the infinity sample space $(X^\infty, \mathcal{F}^\infty)$, then for the parameter $\theta = (\theta_1, \ldots, \theta_q)$ belonging to the set

(1.2) $$\Theta = \Xi^q$$

the corresponding product measure

(1.3) $$P_\theta = \overline{P}_{\theta_1}^\infty \times \ldots \times \overline{P}_{\theta_q}^\infty$$

describes independent sampling from the $q$ populations $(X, \mathcal{F}, \overline{P}_{\theta_j})$, $j = 1, \ldots, q$. We shall consider the situation when

(1.4) $$\emptyset \neq \ \Omega_0 \subset \ \Omega_1 \subset \ \Theta\,,$$

the null hypothesis $H_0$ is that $\theta \in \Omega_0$ and the alternative hypothesis is that $\theta$ is an element of $\Omega_1 - \Omega_0$.

Let $T_u : S \to R$ be a test statistic, $u = 1, 2, \ldots$ and the hypothesis $H_0$ is rejected whenever $T_u$ equals or exceeds a chosen critical value. The smallest level of significance, for which the test rejects $H_0$ if $T_u(s)$ is observed, is the number

(1.5) $$L_u(s) = 1 - G_u(T_u(s))$$

where

$$G_u(t) = \inf \{ F_u(t,\theta);\ \theta \in \Omega_0 \}, \qquad F_u(t,\theta) = P_\theta[T_u < t] \tag{1.6}$$

and is called the level attained by the statistic $T_u$. We suppose that sampling is made from $q$ populations, the sample size of the $u$th sample from the $j$th population is $n_u^{(j)}$, $j = 1, \dots, q$ and that $T_u$ is a function of the $u$th sample, or more precisely, that $T_u$ depends on $s \in S$ through

$$x^{(u)} = ((x_1^{(1)}, \dots, x^{(1)}_{n_u^{(1)}}), \dots, (x_1^{(q)}, \dots, x^{(q)}_{n_u^{(q)}})) \tag{1.7}$$

only, where $(x_1^{(j)}, \dots, x^{(j)}_{n_u^{(j)}})$ is a sample from the $j$th population. Under the alternative the level attained in typical cases tends to zero exponentially fast. If we denote

$$n_u = \sum_{j=1}^{q} n_u^{(j)} \tag{1.8}$$

and if for $\theta \in \Omega_1 - \Omega_0$ the relation

$$\lim_{u\to\infty} \frac{2}{n_u} \log L_u(s) \ = \ -C(\theta) \tag{1.9}$$

holds almost everywhere $P_\theta$, then according to [3] the quantity $C(\theta)$ is called the exact slope of the sequence $\{T_u\}$. In order that we could bound this rate of convergence, we introduce the following assumption.

*(A I) If $u \neq v$, then $n_u^{(j)} \neq n_v^{(j)}$ for some $j$,*

$$\lim_{u\to\infty} n_u = +\infty \quad and \quad \lim_{u\to\infty} \frac{n_u^{(j)}}{n_u} = p_j \in (0,1\rangle \ \ for \ \ j = 1, \dots, q\,. \tag{1.10}$$

If for $\theta = (\theta_1, \dots, \theta_q)$, $\theta^* = (\theta_1^*, \dots, \theta_q^*)$ belonging to $\Theta$ we put

$$K(\theta, \theta^*) = \sum_{j=1}^{q} p_j K(\theta_j, \theta_j^*) \tag{1.11}$$

where $K(\theta_j, \theta_j^*) = K(\overline{P}_{\theta_j}, \overline{P}_{\theta_j^*})$ is the Kullback-Leibler information number, and denote

$$J(\theta) = \inf \{ K(\theta, \theta^*);\ \theta^* \in \Omega_0 \}, \tag{1.12}$$

then a straightforward extension of the result from [10] (cf. also p.29 in [4]) yields that for $\theta \in \Omega_1 - \Omega_0$ the inequality

$$\liminf_{u\geq 1} \frac{2}{n_u} \log L_u(s) \ \geq \ -2J(\theta) \tag{1.13}$$

holds a. e. $P_\theta$, provided that the assumption (A I) is true. Hence if in this situation the exact slope exists, it cannot exceed $2J(\theta)$.

Optimality of the likelihood ratio test statistic in the sense of maximization of the exact slope was in [3] proved under the assumption that the parameter space is finite. Further logical step is to prove optimality by means of compactness conditions. Since in the usual cases the parameter set is not compact (with the exception of the multinomial distribution), Bahadur proved in [2] the optimality under conditions, including existence of a "suitable compactification". However, as observed in [5], p. 149, verification of these assumptions seems to be a formidable task even in the case of testing the hypothesis $\mu_1 = \ldots = \mu_q$ (under the usual assumption of normality and equality of variances). In further research the compactification approach was therefore abandoned. Optimality of a statistic in the Bahadur sense is in [9] proved under conditions, including the assumption, that distribution of the statistic under validity of $\Omega_0$ does not depend on $\theta_0 \in \Omega_0$ and that the ratio $f(t,\theta)/f(t,\theta_0)$ of densities is an increasing function of $t$. Hsieh proved in [7] optimality of a test statistic $T_n$ under the assumption that $\liminf_n T_n \geq J(\theta)$ a.e. $P_\theta$ and that $\exp[-nT_n]$ is under $\Omega_0$ distributed as a product of beta variates. We shall prove optimality of the LR statistic under regularity conditions, which we show to be fulfilled by regular normal distribution, exponential and Laplace distribution, and apply them to the Poisson distribution.

Throughout the paper we shall assume that the probabilities $\{\overline{P}_\gamma;\ \gamma \in \Xi\}$ are defined by means of the densities

$$f(x,\gamma) = \frac{\mathrm{d}\overline{P}_\gamma}{\mathrm{d}\nu}(x) \tag{1.14}$$

where $\nu$ is a $\sigma$-finite measure. We shall use the notation

$$L(x_1,\ldots,x_n,\gamma) = \prod_{i=1}^{n} f(x_i,\gamma)\,, \quad L(x_1,\ldots,x_n,V) = \sup\{\,L(x_1,\ldots,x_n,\gamma);\ \gamma \in V\,\} \tag{1.15}$$

and for $\theta = (\theta_1,\ldots,\theta_q) \in \Theta$ in accordance with (1.7) we put

$$L(x^{(u)},\theta) = \prod_{j=1}^{q} L(x_1^{(j)},\ldots,x_{n_u^{(j)}}^{(j)},\theta_j)\,, \qquad L(x^{(u)},\Omega) = \sup\,\{\,L(x^{(u)},\theta);\ \theta \in \Omega\,\}\,. \tag{1.16}$$

*(A II) $\Xi$ is a metric space, if $q \geq 1$ is an integer and $\Omega \subset \Xi^q$ is either open or a closed set, then $L(x^{(u)},\Omega)$ is a measurable function of $x^{(u)}$, and if $\gamma \in \Xi$, then there exists a set $A_\gamma$ such that*

$$\nu(A_\gamma) = 0 \tag{1.17}$$

*and for every sequence $\{\gamma_n\}_{n=1}^{\infty}$ of elements from $\Xi$ tending to $\gamma$ the equality*

$$\lim_{n\to\infty} f(x,\gamma_n) = f(x,\gamma) \tag{1.18}$$

*holds, whenever $x \in X - A_\gamma$.*

*(A III) Let $\gamma \in \Xi$ and $\eta > 0$ is a real number. One can find measurable sets $A_n \subset X^n$ and a positive integer $N$ with the following properties.*

*1) The inequality*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\gamma(A_n) \leq -\eta \tag{1.19}$$

*holds.*

*2) With the notation (1.15)*

$$\inf \left\{ \frac{1}{n} \log L(x_1, \dots, x_n, \gamma);\ (x_1, \dots, x_n) \in X^n - A_n,\ n \geq N \right\} > -\infty\,. \tag{1.20}$$

*3) If $c$ is a real number, then there exists a non-empty compact set $\Gamma_c \subset \Xi$ such that*

$$\sup \left\{ \frac{1}{n} \log L(x_1, \dots, x_n, \Xi - \Gamma_c);\ (x_1, \dots, x_n) \in X^n - A_n,\ n \geq N \right\} \leq c\,. \tag{1.21}$$

*(A IV) If $\gamma \in \Xi$, then there exists a real number $\delta > 0$ such that with the notation*

$$V = V(\gamma, \delta) = \{\gamma^* \in \Xi;\ \rho(\gamma^*, \gamma) < \delta\} \tag{1.22}$$

*the inequality*

$$\int L(x, V)\, d\nu(x) < +\infty \tag{1.23}$$

*holds.*

*(A V) If $\gamma \in \Xi$ and $\eta > 0$ is a real number, then there exists a real number $\varepsilon > 0$ such that*

$$\limsup_{n \geq 1} \frac{1}{n} \log P_\gamma \left[ \frac{1}{n} \log \frac{L(x_1, \dots, x_n, \Xi)}{L(x_1, \dots, x_n, \gamma)} \geq \varepsilon \right] \leq -\eta\,. \tag{1.24}$$

*(A VI) If $\gamma$, $\gamma_n$, $n = 1, 2, \dots$ belong to $\Xi$, then*

$$\lim_{n \to \infty} K(\gamma, \gamma_n) = 0 \tag{1.25}$$

*implies*

$$\lim_{n \to \infty} \gamma_n = \gamma\,. \tag{1.26}$$

*(A VII) If (A I) holds, then there exists a number $N_1$, a constant $c_1$ and a point $\vartheta \in \Theta$ such that*

$$\sup \left\{ P_\theta \left[ 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \theta)} \geq t \right];\ \theta \in \Theta \right\} = P_\vartheta \left[ 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \vartheta)} \geq t \right] \tag{1.27}$$

*for all $u \geq N_1$ and $t \geq c_1$.*

**Theorem 1.1.** *Let us assume that the assumptions (A I) - (A IV) hold, the inclusions (1.4) are valid, $\Omega_0$ is a closed set and $\Omega_1$ is either closed or open. Let us put*

$$T_u(s) = 2\, \log \frac{L(x^{(u)}, \Omega_1)}{L(x^{(u)}, \Omega_0)} \tag{1.28}$$

*where* log *denotes logarithm to the base e, the vector (1.7) corresponds to $s \in S$, $0/0 = 1$ and $a/0 = +\infty$ for $a > 0$. Let $\theta \in \Omega_1 - \Omega_0$.*

*(I) The relation*

$$\lim_{u\to\infty} \frac{T_u(s)}{n_u} = 2J(\theta) \tag{1.29}$$

*holds a.e. $P_\theta$.*

*(II) If (A V) - (A VII) hold, then (1.9) holds a.e. $P_\theta$ with*

$$C(\theta) = 2J(\theta) \tag{1.30}$$

*and therefore the likelihood ratio test statistic (1.28) is optimal in the sense of exact slopes.*

If the regularity conditions are modified, we get a version of Theorem 1.1, not imposing conditions on closedness of $\Omega_0$, $\Omega_1$.

*(AR II) $\Xi$ is a separable metric space and the function $f(x,.)$ is continuous for each $x \in X$.*

*(AR VI) The function $\varphi(\gamma^*) = K(\overline{P}_\gamma, \overline{P}_{\gamma^*})$ is continuous on $\Xi$ for all $\gamma \in \Xi$, and if $\gamma$, $\gamma_n$, $n = 1, 2, \dots$ belong to $\Xi$, then (1.25) holds if and only if (1.26) is true.*

**Theorem 1.2.** *Let the assumptions (A I), (AR II), (A III), (A IV) and (AR VI) hold, and $T_u$ is the statistic (1.28) determined by the sets (1.4). Let us assume that $\theta \in \Omega_1 - \Omega_0$.*

*(I) The relation (1.29) holds a.e. $P_\theta$.*

*(II) If also (A V) and (A VII) hold, then (1.9) holds a.e. $P_\theta$ with $C(\theta) = 2J(\theta)$ and the test statistic (1.28) is optimal in the sense of exact slopes.*

Maximization of $C(\theta)$ is a nice property, because the greater the exact slope is, the smaller the level attained tends to be, and the more an experimenter can be convinced that the rejected hypothesis is indeed false. However, finding value of the level attained can turn out to be a difficult process, either if the function $G_u$ defined by (1.6) is complicated, or when it is even unknown (this occurs in Examples 1 and 6 in the 2nd part of the paper). If in the last case at least the limiting distribution is known, the test can be performed by comparing $T_u$ with a quantile of the function

$$G(t) = \inf_{\theta\in\Omega_0} \lim_{u\to\infty} F_u(t,\theta). \tag{1.31}$$

As a measure of evidence against the null hypothesis serves now the approximate level attained

$$L_u^{(a)}(s) = 1 - G(T_u(s)) \tag{1.32}$$

whose asymptotic behaviour, when a non-nul $\theta$ obtains, can be characterized by the approximate slope. This quantity, introduced in [1] and [3], is a number $C^{(a)}(\theta)$, satisfying with probability 1 the equality

$$\lim_{u\to\infty} \frac{2}{n_u} \log L_u^{(a)}(s) = -C^{(a)}(\theta)\,. \tag{1.33}$$

A relation between the approximate and the exact slope is a topic of the following theorem, in which by a finite mixture $G$ of chi-square distributions we understand the function

$$G(t) = \sum_{j=1}^{m} \alpha_j P[\chi^2_{d_j} < t]\,, \tag{1.34}$$

where $m$ is a positive integer, $\chi^2_{d_j}$ is a random variable which has chi-square distribution with $d_j$ degrees of freedom (by convention $P[\chi^2_0 = 0] = 1$), $\max_j d_j > 0$, the numbers $\alpha_1, \ldots, \alpha_m$ are positive and their sum equals 1.

**Theorem 1.3.** *Let $T_u$ be the statistic (1.28) determined by the sets (1.4). Let (1.29) holds a.e. $P_\theta$, the assumption (A VI) and the equality (1.30) are true and $\theta \in \Omega_1 - \overline{\Omega}_0$.*

*(I) If the function (1.31) is such that*

$$\log[1 - G(t)] = -\frac{1}{2}t[1 + o(1)] \tag{1.35}$$

*where $\lim_{t\to+\infty} o(1) = 0$, then the approximate slope exists and*

$$C^{(a)}(\theta) = C(\theta)\,. \tag{1.36}$$

*(II) If the function (1.31) is a finite mixture of chi-square distributions, then the approximate slope exists and (1.36) holds.*

Since according to [8], p. 173

$$P\left[\chi^2_{2\nu} \geq x\right] = \exp(-x/2) \sum_{j=0}^{\nu-1} (x/2)^j / j!$$

$$P\left[\chi^2_{2\nu-1} \geq x\right] = \exp(-x/2) \sum_{j=0}^{\nu-2} (x/2)^{j+0.5} / \Gamma(j + \frac{3}{2}) + 2\left[1 - \Phi(x^{1/2})\right]$$

and the value of the standard normal distribution function can be determined by means of a suitable approximation with a good precision (cf. [8], pp. 53-57), the value of (1.34) can be computed even with a desk calculator. Hence Theorem 1.3 provides a basis for the approximation

$$L_u(s) \doteq 1 - G(T_u(s))$$

by means of a finite mixture of chi-squares, which is numerically not too difficult to handle.

## 2. PROOFS OF ASSERTIONS FROM SECTION 1

**Lemma 2.1.** *In $\{n_u\}_{u=1}^{\infty}$ is a sequence of integers and $\{\, a(j, n_u);\ j = 1, \ldots, k\ \ u = 1, 2, \ldots \}$ are non-negative numbers, then*

$$\limsup_{u\to\infty} \frac{1}{n_u} \log[\sum_{j=1}^{k} a(j, n_u)] = \max_{j=1,\ldots,k}\ \limsup_{u\to\infty} \frac{1}{n_u} \log a(j, n_u) \tag{2.1}$$

*whenever $k$ is a positive integer and $\lim_{u\to\infty} n_u = +\infty$.*

The proof can be simply carried out by means of the inequalities

$$\log a(j_0, n_u) \le \log[\sum_{j=1}^{k} a(j, n_u)] \le \log[\, k \max_j a(j, n_u)]$$

and is therefore omitted.

**Lemma 2.2.** *Let us assume that (A I) holds and $L(x^{(u)}, \Omega)$ is a measurable function of $x^{(u)}$ whenever $\Omega$ is either open or closed subset of the metric space $\Xi$. Let $\theta$ be an arbitrary but fixed element of $\Theta$.*

*(I) If (A II) and (A IV) hold and $\Gamma \subset \Theta$ is a non-empty compact set, then a.e. $P_\theta$*

$$\lim_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \Gamma)}{L(x^{(u)}, \theta)} = -J(\theta, \Gamma)\,, \tag{2.2}$$

*where (cf. (1.11))*

$$J(\theta, \Gamma) = \inf\{\, K(\theta, \theta^*);\ \theta^* \in \Gamma\,\}\,. \tag{2.3}$$

*(II) Let the assumptions (A II) - (A IV) be valid. Then*

$$\lim_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \theta)} = 0 \tag{2.4}$$

*a.e. $P_\theta$, and if $M > 0$ is a real number, then there exists a non-empty compact set $\Gamma \subset \Theta$ such that a.e. $P_\theta$*

$$\limsup_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \Theta - \Gamma)}{L(x^{(u)}, \theta)} \le -M\,. \tag{2.5}$$

*(III) If the assumptions (A III) and (A V) are true and $\eta > 0$ is a positive number, then there exists a non-empty compact set $\Gamma \subset \Theta$ such that*

$$\limsup_{u\to\infty} \frac{1}{n_u} \log P_\theta[\, L(x^{(u)}, \Theta - \Gamma) \ge L(x^{(u)}, \theta)\,] \le -\eta\,. \tag{2.6}$$

P r o o f. (I) If $\theta^* \in \Theta$ and $V_j$ is a neighbourhood of $\theta_j^*$ for which (1.23) holds, then with the notation $y^+ = \max\{y, 0\}$ by means of $\log x < x$ we get

$$\int \left[\log \frac{L(x, V_j)}{L(x, \theta_j)}\right]^+ \mathrm{d}P_{\theta_j}(x) \;\le\; \int L(x, V_j)\,\mathrm{d}\nu(x) \;<\; +\infty\,.$$

Hence the function $\log[L(x, V_j)/L(x, \theta_j)]$ is $P_{\theta_j}$ integrable with $-\infty$ as a possible value of the integral. Making use of (A II) and the monotonne convergence theorem (in the "almost everywhere" sense), we get that (cf. (1.22)) with the notation $V_\delta = V(\theta_j^*, \delta)$

$$\lim_{\delta\to 0^+} \int \left[\log \frac{L(x, V_\delta)}{L(x, \theta_j)}\right] \mathrm{d}P_{\theta_j}(x) = -K(\theta_j, \theta_j^*)\,. \tag{2.7}$$

In proving (2.2) we now proceed similarly as in the proof of Lemma 4 in [2]. If

$$m < J(\theta, \Gamma) \tag{2.8}$$

then compactness of $\Gamma$ and (2.7) imply that there exist finitely many open subsets

$$W_i = V_1^{(i)} \times \ldots \times V_q^{(i)}\,, \qquad i = 1, \ldots, k\,, \tag{2.9}$$

of $\Theta$ such that

$$\Gamma \subset \bigcup_{i=1}^{k} W_i \tag{2.10}$$

and with the notation (1.10) for $i = 1, \ldots, k$

$$\sum_{j=1}^{q} p_j \int \log \frac{L(x, V_j^{(i)})}{f(x, \theta_j)}\, \mathrm{d}P_{\theta_j}(x) < -m\,. \tag{2.11}$$

Since the number $m$ in (2.8) is arbitrary, from (2.9) - (2.11) and the law of large numbers we easily obtain that a.e. $P_\theta$

$$\limsup_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \Gamma)}{L(x^{(u)}, \theta)} \le -J(\theta, \Gamma)\,. \tag{2.12}$$

But if $\theta^* \in \Gamma$, then according to the law of large numbers a.e. $P_\theta$

$$\liminf_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \Gamma)}{L(x^{(u)}, \theta)} \ge \liminf_{u\to\infty} \frac{1}{n_u} \log \frac{L(x^{(u)}, \theta^*)}{L(x^{(u)}, \theta)} = -K(\theta, \theta^*)\,.$$

Letting $K(\theta, \theta^*) \to J(\theta, \Gamma)$ and taking into account (2.12) we get (2.2).

(II) Let us put in (A III) the number $\eta = 1$ and

$$c = \inf\left\{ \frac{1}{n} \log L(x_1, \ldots, x_n, \gamma);\ (x_1, \ldots, x_n) \in X^n - A_n,\ n \ge N \right\} - M\,.$$

Since $\sum_{n=N}^{\infty} P_\gamma(A_n) < +\infty$, the set $A = \bigcap_{m=N}^{\infty} \bigcup_{n=m}^{\infty} \tilde{A}_n$, where $\tilde{A}_n = \{x^\infty \in X^\infty; (x_1, \ldots$ $\ldots, x_n) \in A_n\}$, has $P_\gamma$ probability zero, and if $x^\infty \notin A$, then

$$\frac{1}{n} \log \frac{L(x_1, \ldots, x_n, \Xi - \Gamma_c)}{L(x_1, \ldots, x_n, \gamma)} \le -M \tag{2.13}$$

for all $n \ge m(x^\infty)$. Making use of (I) and (2.13) we get that a.e. $P_\gamma$

$$\limsup_{n\ge 1} \frac{1}{n} \log \frac{L(x_1, \ldots, x_n, \Xi)}{L(x1, \ldots, x_n, \gamma)} \le \max\{-M, -J(\gamma, \Gamma_c)\} \le 0$$

and the assertion is proved for $q = 1$.

Let $q > 1$. Validity of (2.4) follows from its validity in the case of one sample. If $\Gamma_j \subset \Xi$ is the non-empty compact set for which (2.5) holds with $q = 1$, $\theta = \theta_j$ and $M_j = M/p_j$, then denoting

$$\Gamma = \Gamma_1 \times \ldots \times \Gamma_q\,, \tag{2.14}$$

utilizing subaditivity of limes superior, (2.4), and making use of the obvious inequality

(2.15)
$$\frac{1}{n_u}\log\frac{L(x^{(u)},\Theta-\Gamma)}{L(x^{(u)},\theta)} \le \max_{j=1,=\dots,q}\frac{1}{n_u}\log\frac{L(x^{(u)},\Xi\times\ldots\times\Xi\times(\Xi-\Gamma_j)\times\Xi\times\ldots\times\Xi)}{L(x^{(u)},\theta)}$$
we get validity of (2.5) a.e. $P_\theta$.

(III) If $j\in\{1,\ldots,q\}$, then according to (A III) and (A V) there exist measurable sets $\{A_n^{(j)}\}_{n=1}^{\infty}$ and a real number $\varepsilon_j>0$ such that both (1.19) and (1.24) hold with $\gamma=\theta_j$, $\tilde\eta=\eta/p_j$ and that
$$\alpha_j=\inf\left\{\frac{1}{n}\log L(x_1,\ldots,x_n,\theta_j);\ (x_1,\ldots,x_n)\in X^n-A_n^{(j)},\ n\ge N\right\}$$
is a real number. Let us denote $\varepsilon=\sum_{j=1}^{q}\varepsilon_j$ and choose a non-empty compact subset $\Gamma_j=\Gamma_c$ of $\Xi$ such that (1.21) holds with $A_n=A_n^{(j)}$ and
$$c=c_j=\alpha_j-\frac{2}{p_j}\varepsilon-1\,.$$
Since the set (2.14) is compact, it is sufficient to prove that it satisfies (2.6). With the notation
$$y_u^{(i)}=(x_1^{(i)},\ldots,x_{n_u^{(i)}}^{(i)}) \tag{2.16}$$
the inequality
$$P_\theta\left[\log\frac{L(x^{(u)},\Xi\times\ldots\times\Xi\times(\Xi-\Gamma_j)\times\Xi\times\ldots\times\Xi)}{L(x^{(u)},\theta)}\ge 0\right]\le \tag{2.17}$$
$$\le\sum_{i\ne j}P_{\theta_i}\left[\frac{1}{n_u}\log\frac{L(y_u^{(i)},\Xi)}{L(y_u^{(i)},\theta_i)}\ge\varepsilon_i\right]+\delta(u,j)\,,$$
$$\delta(u,j)=P_{\theta_j}\left[\frac{1}{n_u}\log\frac{L(y_u^{(j)},\Xi-\Gamma_j)}{L(y_u^{(j)},\theta_j)}>-\varepsilon\right]$$
obviously holds, and from the choice of $c_j$
$$\limsup_{u\to\infty}\frac{1}{n_u}\log\delta(u,j)\ \le\ \limsup_{u\to\infty}\frac{1}{n_u}\log P_\theta\left[A_{n_u^{(j)}}^{(j)}\right]\ \le\ -\eta\,.$$
Hence making use of (2.15), (2.17) and Lemma 2.1 we get (2.6). □

P r o o f o f T h e o r e m 1.1. (I) If $\theta^*\in\Omega_0$, then making use of Lemma 2.2 (II) and the law of large numbers we get that a.e. $P_\theta$
$$\limsup_{u\to\infty}\frac{1}{n_u}T_u(s)\ \le\ \limsup_{u\to\infty}\frac{2}{n_u}\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\theta^*)}=$$
$$=\limsup_{u\to\infty}\frac{2}{n_u}\log\frac{L(x^{(u)},\theta)}{L(x^{(u)},\theta^*)}=2K(\theta,\theta^*)\,,$$
which means that
$$\limsup_{u\to\infty}\frac{1}{n_u}T_u(s)\ \le\ 2J(\theta) \tag{2.18}$$

a.e. $P_\theta$. Let a real number

(2.19) $$M < 2J(\theta)$$

and $\Gamma \subset \Theta$ be a non-empty compact set, satisfying (2.5) a.e. $P_\theta$ with $\tilde{M} = M/2$. If the set $\Omega_0 - \Gamma$ is non-empty, then a.e. $P_\theta$

(2.20) $$\liminf_{u\to\infty} \frac{2}{n_u} \log \frac{L(x^{(u)},\theta)}{L(x^{(u)},\Omega_0 - \Gamma)} \geq \liminf_{u\to\infty} \frac{2}{n_u} \log \frac{L(x^{(u)},\theta)}{L(x^{(u)},\Theta - \Gamma)} \geq M\,.$$

If $\Omega_0 \cap \Gamma$ is a non-empty set, then taking into account compactness of $\Omega_0 \cap \Gamma$ and Lemma 2.2 (I) we see that a.e. $P_\theta$

(2.21) $$\liminf_{u\to\infty} \frac{2}{n_u} \log \frac{L(x^{(u)},\theta)}{L(x^{(u)},\Omega_0 \cap \Gamma)} = 2J(\theta, \Omega_0 \cap \Gamma) > M\,.$$

Since

$$T_u(x^{(u)}) \geq \min\left\{2\log\frac{L(x^{(u)},\theta)}{L(x^{(u)},\Omega_0 - \Gamma)}\,,\; 2\log\frac{L(x^{(u)},\theta)}{L(x^{(u)},\Omega_0 \cap \Gamma)}\right\}$$

the assertion (I) can be easily proved by means of (2.18) - (2.21).

(II) Since $\theta$ does not belong to the closed set $\Omega_0$, according to (A VI)

(2.22) $$J(\theta) > 0\,.$$

Let $\vartheta$ be the parameter from (A VII) and

(2.23) $$L_u^*(t) = P_\vartheta\left[2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\vartheta)} \geq t\right].$$

It is obvious from (I), (2.22), (A VII), (1.5) and (1.28) that a.e. $P_\theta$ for all $u \geq U(s)$

$$L_u(s) \leq L_u^*(\,T_u(s)\,)\,.$$

Hence taking into account (I), (2.22) and (1.13) we see that it is sufficient to prove the following lemma.

**Lemma 2.3.** *If $\{t_u\}_{u=1}^{\infty}$ is a sequence of elements from $\langle 0, +\infty\rangle$ and*

(2.24) $$\lim_{u\to\infty} \frac{t_u}{n_u} = t > 0$$

*then with the notation (2.23)*

(2.25) $$\limsup_{u\to\infty} \frac{1}{n_u} \log L_u^*(t_u) \leq -t/2\,.$$

Proof. If $\eta < t/2$ is a real number and $\Gamma \subset \Theta$ is the non-empty compact set satisfying (2.6) with $\theta = \vartheta$, then taking into account Lemma 2.1 we get the inequality

(2.26) $$\limsup_{u\to\infty} \frac{1}{n_u} \log L_u^*(t_u) \leq \max\{-\eta\,,\,\delta\}\,,$$

where

(2.27) $$\delta = \limsup_{u\to\infty} \frac{1}{n_u} \log P_\vartheta \left[ 2\log \frac{L(x^{(u)},\Gamma)}{L(x^{(u)},\vartheta)} \geq t_u \right] .$$

Let $\varepsilon > 0$ be a fixed real number. Making use of (A II), (A IV), the monotone convergence theorem (similarly as in proving (2.7)) and the compactness of $\Gamma$, we see that there exist finitely many sets $W_1, \ldots, W_k$ such that (2.9), (2.10) hold and

(2.28) $$\int L(x, V_j^{(i)})\,\mathrm{d}\nu(x) \;\leq\; 1+\varepsilon\,, \qquad i=1,\ldots,k\,, \qquad j=1,\ldots,q\,.$$

Let the integer $i \in \{1,\ldots,k\}$ be fixed. If we put

$$y_j(x) = \log \frac{L(x, V_j^{(i)})}{f(x,\vartheta_j)}\,, \qquad j=1,\ldots,q\,,$$

then making use of the Markov inequality and (2.28) we get that for real $t_u$

$$P_\vartheta \left[ 2\log \frac{L(x^{(u)},W_i)}{L(x^{(u)},\vartheta)} \geq t_u \right] \;\leq$$
$$\leq \int \exp[\sum_{j=1}^{q} \sum_{\alpha=1}^{n_u^{(j)}} y_j(x_\alpha^{(j)}) - \frac{1}{2} t_u] \prod_{j=1}^{q} \prod_{\alpha=1}^{n_u^{(j)}} f(x_\alpha^{(j)},\vartheta_j) \prod_{j=1}^{q} \prod_{\alpha=1}^{n_u^{(j)}} \mathrm{d}\nu(x_\alpha^{(j)}) \;\leq$$
$$\leq \exp[-\frac{1}{2} t_u](1+\varepsilon)^{n_u}$$

and the resulting inequality is true also if $t_u = +\infty$. This together with (2.10) yields

$$P_\vartheta \left[ 2\log \frac{L(x^{(u)},\Gamma)}{L(x^{(u)},\vartheta)} \geq t_u \right] \;\leq\; k\,\exp[-\frac{1}{2} t_u](1+\varepsilon)^{n_u}\,.$$

Hence the quantity (2.27) satisfies the inequality

$$\delta \leq -\frac{1}{2} t + \log(1+\varepsilon)$$

and letting $\varepsilon \to 0$ we obtain from (2.26) easily (2.25). □

Proof of Theorem 1.2. Let $\theta$ belong to the closure $\overline{\Omega}_0$ of the set $\Omega_0$. This according to (AR VI) means that

(2.29) $$J(\theta) = 0\,.$$

But if $\theta^* \in \Omega_0$, then making use of (2.4) we get that a.e. $P_\theta$

$$\limsup_{u\to\infty} \frac{1}{n_u} T_u(s) \;\leq\; \limsup_{u\to\infty} \frac{2}{n_u} \log \frac{L(x^{(u)},\theta)}{L(x^{(u)},\theta^*)} \;=\; 2K(\theta,\theta^*)\,,$$

and since $T_u(s) \geq 0$, (1.29) follows from (2.29). From (1.5) we obtain that

$$\limsup_{u\to\infty} \frac{1}{n_u} \log L_u(T_u(s)) \;\leq\; 0 \tag{2.30}$$

and combining (2.29) with (1.13) yields (1.9) and (1.30).

If $\theta \notin \overline{\Omega}_0$, then taking into account (AR II) we get the equality

$$T_u(s) = 2 \log \frac{L(x^{(u)}, \overline{\Omega}_1)}{L(x^{(u)}, \overline{\Omega}_0)} \tag{2.31}$$

and (1.29) follows from Theorem 1.1. Making use of the Scheffé theorem (cf. [11], section 2.c) we see that

$$L_u(s) = \overline{L}_u(s)$$

where $\overline{L}_u$ is the level attained by the statistic (2.31) for testing $\overline{\Omega}_0$ against $\overline{\Omega}_1$, and (II) follows from Theorem 1.1 (II).

P r o o f  o f  T h e o r e m  1.3. Since $\theta \notin \overline{\Omega}_0$, (2.22) holds and according to the assumptions

$$\lim_{u\to\infty} T_u(s) = +\infty$$

a.e. $P_\theta$. Combining this with (1.35) and (1.29) we obtain that the first assertion is true. Now, if $F_k$ denotes the distribution function of the chi-square distribution with $k$ degrees of freedom, then for $k > 0$ according to Lemma 3.2 in [6]

$$\log[1 - F_k(t)] = -\frac{1}{2} t[1 + o(1)] \tag{2.32}$$

where $\lim_{t\to+\infty} o(1) = 0$. Since the mixture (1.34) satisfies the inequality

$$\alpha(1 - F_1(t)) \leq 1 - G(t) \leq \alpha(1 - F_d(t))$$

where $d = \max_j d_j$ and $\alpha$ is the sum of the coefficients $\alpha_j$ for which $d_j > 0$, the condition (1.35) follows from (2.32). □

We remark that the assumption of possible discontinuity of densities necessitates postulating measurability of $L(x^{(u)}, \Omega)$. To verify this condition the following lemma appears to be useful.

**Lemma 2.4.** *Let $\Xi$ be an open subset of $R^m$ and $\{\, f(x, \gamma);\ \gamma \in \Xi \,\}$ be the densities (1.14). Let us assume that for each $\gamma \in \Xi$ there exist $\{\gamma_k\}_{k=1}^{\infty}$ from $\tilde{\Xi} = \{\, \gamma \in \Xi;$ all coordinates of $\gamma$ are rational numbers$\,\}$ such that*

$$\lim_{k\to\infty} \gamma_k = \gamma \tag{2.33}$$

*and*

$$\lim_{k\to\infty} f(x, \gamma_k) = f(x, \gamma) \tag{2.34}$$

*for all $x \in X$. If (2.33) implies that for each $x \in X$*

$$\limsup_{k\to\infty} f(x,\gamma_k) \le f(x,\gamma) < +\infty \tag{2.35}$$

*then $L(x^{(u)}, \Theta \cap C)$ is a measurable function of $x^{(u)}$ whenever the set $C \subset R^{mq}$ is either open or closed.*

*Proof.* If the set $C$ is open, then $L(x^{(u)}, \Theta \cap C) = L(x^{(u)}, C \cap \tilde{\Xi}^q)$ where the last function is obviously measurable. If the set $C \subset \Theta$ is compact, $\varepsilon > 0$ and $C_\varepsilon = \{\, \theta^*;\ \max_i \rho(\theta_i, \theta_i^*) < \varepsilon$ *for some* $\theta \in C \,\}$, then according to (2.35)

$$\lim_{\varepsilon\to 0^+} L(x^{(u)}, \Theta \cap C_\varepsilon) = L(x^{(u)}, \Theta \cap C)$$

and since the sets $C_\varepsilon$ are open, measurability of $L(.,C)$ follows. Since the sets

$$W_k = \{\, \theta^* \in \Theta;\ \|\theta^*\| \le k\,,\ \ \rho(\theta^*, R^{mq} - \Theta) \ge k^{-1} \,\}$$

form an increasing sequence of compact sets whose union equals $\Theta$, the lemma is proved. □

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