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A QUANTILE GOODNESS-OF-FIT TEST APPLICABLE TO DISTRIBUTIONS WITH NON-DIFFERENTIABLE DENSITIES

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Asymptotic distribution of the random vector of differences between theoretical probabilities and their estimates, based on the sample quantiles and on an estimate of the unknown parameter, is derived in a setting not requiring differentiability of the densities. By means of this result asymptotically chi-square distributed goodness-of-fit test statistics are constructed for the exponential distribution and for the Laplace distribution.

1. INTRODUCTION

In this paper we deal with testing goodness of fit for probabilities defined on the Borel subsets of the real line. Tests of the null hypothesis that the underlying distribution P belongs to the given family $\{P_\theta;\, \theta \in \Theta\}$ of distributions can be constructed in various ways. Not aiming to make a detailed list of approaches or results, let us mention some of them. One possibility is comparison of the difference between the empirical and the theoretical distribution function with theoretical critical values (the Kolmogorov-Smirnov test). Another approach consists in employing a test statistic utilizing some typical properties of the null class of distributions, e.g., their shape. Here one can mention the omnibus D'Agostino test from [5], the Shapiro-Wilk test (cf. [17]) for normality, or the test for two-parameter exponential distribution described in [11] and [18]. A test procedure based on the empirical characteristic function is described in [10]. The most classical approach for testing the null hypothesis when $\Theta \subset R^m$ is an open set, is the minimum chi-square method, described in [4], or in [1], pp. 196-201. We recall that this method uses a partition of the sample space into $k+1$ disjoint cells $C_1, \ldots, C_{k+1}$ and that the test statistic based on the random sample $x_1, \ldots, x_n$ is given by

$$T_n = \sum_{i=1}^{k+1} \frac{[X_i - nP_{\tilde\theta}(C_i)]^2}{nP_{\tilde\theta}(C_i)} = n \sum_{i=1}^{k+1} \frac{[\hat P(C_i) - P_{\tilde\theta}(C_i)]^2}{P_{\tilde\theta}(C_i)} \,, \tag{1.1}$$

where $X_1, \ldots, X_{k+1}$ are the observed cell frequencies, $\hat P(C_i) = \frac{X_i}{n}$ is estimate of the probability $P_\theta(C_i)$ based on the cell frequencies, and the estimate $\tilde\theta = \tilde\theta(X_1, \ldots, X_{k+1})$ is computed from the equations

$$\sum_{i=1}^{k+1} \frac{X_i}{P_\theta(C_i)} \frac{\partial P_\theta(C_i)}{\partial \theta_j} = 0\,, \qquad j = 1, \ldots, m. \tag{1.2}$$

If certain regularity conditions on the probabilities $(\,P_\theta(C_1),\dots,P_\theta(C_{k+1})\,)$ are fulfilled, then

$$\mathcal{L}(T_n) \longrightarrow \chi^2_{k-m} \tag{1.3}$$

as $n \to \infty$, provided that $x_1,\dots,x_n$ is a random sample from the distribution P_θ and $\tilde\theta$ converges to θ in probability. It seems to be logical to prefer a method of this type when subject of the statistical interest is probability of an interval on the real line or when some frequencies have to be interpreted, because such a method uses a fit of some kind of histogram with some theoretical model. However, this particular method has some disadvantages, when applied to certain probability families. The first one is that for some families an explicit formula for solution of (1.2) is not available. Further difficulty lies in the assumption that the partitioning $C_1,\dots,C_{k+1}$ is unrelated to the random sample $x_1,\dots,x_n$, which in the practice is almost never the case, and partitioning based on the observed values $x_1,\dots,x_n$ can spoil the weak convergence (1.3).

These difficulties gave rise to the approach, when the estimate in (1.1) is not computed from the equations (1.2) and the partitioning is constructed by means of $\hat\theta_n = \hat\theta_n(x_1,\dots,x_n)$, i.e., in dependence on the random sample (e.g., [19], [20] and [6], [12]). However, in these quoted papers the limiting distribution of the test statistics is no longer a chi-square distribution and depends on the way in which the partition was constructed by means of $\hat\theta_n$.

This happens neither in [3] nor in [2]. According to Theorem 4 in [3], or according to Theorem 3 and Theorem 4 in [2], under certain regularity conditions the statistics

$$\chi^2_Q = n\sum_{i=1}^{k+1}\Big[F(x_n^{(n_i)},\hat\theta_n) - F(x_n^{(n_{i-1})},\hat\theta_n) - (p_i - p_{i-1})\Big]^2\Big[p_i - p_{i-1}\Big]^{-1} \tag{1.4}$$

are asymptotically chi-square distributed. Here $x_n^{(j)}$ denotes the j-th order statistic of the random sample $x_1,\dots,x_n$, $n_i = [np_i]+1$ for $i = 1,\dots,k$, $p_0 = 0 < p_1 < \dots < p_k < p_{k+1} = 1$ are fixed numbers and $\hat\theta_n = \hat\theta_n(x_1,\dots,x_n)$ is the estimate minimizing χ^2_Q. However, finding an explicit formula for such an estimate is usually an intractable problem.

The difficulties with the limiting distribution do not arise in the approach used in [13] on p. 591 and in [14] either. In the mentioned papers asymptotic distribution of the test statistics

$$T_n = \tilde{\mathbf{\Delta}}^T(\hat\theta_n)\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{\Delta}}(\hat\theta_n) \tag{1.5}$$

is shown to be chi-square distribution with $k-1$ degrees of freedom. In this notation $\tilde{\mathbf{B}}$ is the asymptotic covariance matrix of the random vector $\tilde{\mathbf{\Delta}}(\hat\theta_n)$, consisting of the differences $\sqrt{n}\Big[P_{\hat\theta_n}(\,\langle x_n^{(n_{i-1})}, x_n^{(n_i)})\,) - (p_i - p_{i-1})\Big]$, $\hat\theta_n = \hat\theta_n(x_1,\dots,x_n)$ is an efficient estimate (in the Rao sense) of the unknown parameter, and $x_n^{(n_i)}$ has the same sense as in the previous case. Validity of this result is established under regularity assumptions closely related to those from [12].

The aim of the paper is to present a set of conditions which can be applied to some probability classes, not fulfilling the assumptions on which proofs of the asymptotic behaviour of the quadratic test statistics (1.1), (1.4) or (1.5) are based, or not being tractable in the models substantiating the mentioned procedures, but still allowing to preserve the simplicity of the asymptotic distribution.

Indeed, it is required in [6] that the density functions $f(x, \theta)$ are differentiable with respect to the parameters, and the resulting asymptotic distribution of the used test statistics is a weighted sum of chi-squares with weights determined by the choice of the partitioning. In contradistinction to this, no assumption on differentiability of the density function is included in *(C 1) - (C 5)* whatsoever and the resulting asymptotic distribution is a chi-square distribution with degrees of freedom determined only by the number of points of the quantile partitioning and not by particular values of the quantiles; a similar situation occurs when our setting is compared with that in [19] and [20].

The assumptions employed in [13] on p. 591, in [14] or in [12] not only require differentiability of the density functions with respect to the parameter and exchangeability of the integration and differentiation sign, but they postulate that the estimate of the unknown parameter has a non-singular asymptotic covariance matrix as well, which are constraints not included in *(C 1) - (C 5)* in the next section. This enlarges the range of cases in which the quantile test can be applied, but it causes some difficulties with regularity of the asymptotic covariance matrix. As it is explained also in the discussion following Theorem 2.2, the situation can be handled by means of Theorem 2.1(III). We remark that if instead of the differentiability condition one would consider a differentiability a.e., then the exchangeability condition would retain its sense, but as one can easily see, in the case of the exponential densities (2.31) which satisfy our regularity conditions, neither this modified exchangeability assumption is fulfilled.

Finally we point out to the fact that the fixed partition approach used in deriving the statistic (1.1) is inapplicable to classes of probabilities with variable lower bounds. Indeed, let $-\infty = c_0 < c_1 < \ldots < c_k < c_{k+1} = +\infty$ be fixed real numbers and $C_i = (c_{i-1}, c_i\rangle$, $i = 1, \ldots, k+1$ denotes partitioning of the real line into disjoint cells. If the sample $x_1, \ldots, x_n$ is taken from distribution with density (2.31) and $\mu \geq c_k$, then with probability 1 the observed cell frequencies $X_1 = \ldots = X_k = 0$, $X_{k+1} = n$ and the formulas (1.1), (1.2) do not yield any useful result.

2. MAIN RESULTS

Let us assume that $\{P_\theta;\, \theta \in \Theta\}$ is a family of probabilities defined on the real line and $x_1, \ldots, x_n$ is a random sample from a distribution P on $(R^1, \mathcal{B}^1)$. Let

$$\hat{F}_n(t) = \frac{1}{n} \sum_{j=1}^{n} \chi_{(-\infty,t\rangle}(x_j) \tag{2.1}$$

denote the sample distribution function. For $0 < p < 1$ let

$$\hat{\xi}_{p,n} = \inf \{t;\ \hat{F}_n(t) \geq p\} \tag{2.2}$$

denote the p-th sample quantile and

$$\xi(p, \theta) = \inf \{t;\ F(t, \theta) \geq p\} \tag{2.3}$$

the p-th quantile of the distribution function

$$F(t, \theta) = P_\theta(\,(-\infty, t\rangle\,)\ . \tag{2.4}$$

Let us choose an integer $k \geq 1$ and real numbers

$$0 < p_1 < \ldots < p_k < 1\ . \tag{2.5}$$

For the random sample $x_1, \ldots, x_n$ and for an estimate $\hat{\theta}_n = \hat{\theta}_n(x_1, \ldots, x_n)$ of the unknown parameter $\theta \in \Theta$ we compute the vector of the differences

$$\mathbf{\Delta_n} = \begin{pmatrix} F(\hat{\xi}_{p_1,n}\,, \hat{\theta}_n) - p_1 \\ F(\hat{\xi}_{p_2,n}\,, \hat{\theta}_n) - p_2 \\ \vdots \\ F(\hat{\xi}_{p_k,n}\,, \hat{\theta}_n) - p_k \end{pmatrix}\ . \tag{2.6}$$

In the proof of the assertion of Theorem 2.1 on asymptotic distribution of (2.6) we shall use the Bahadur representation of the sample quantiles in its simplified version from [7]. This will be carried out in a setting based on the following regularity conditions.

(C 1) $\Theta \subset R^m$ is an open set and the probabilities $\{P_\theta;\ \theta \in \Theta\}$ are defined by means of the densities $\{f(x, \theta);\ \theta \in \Theta\}$ with respect to the Lebesque measure on the real line.

(C 2) The inequality $f(\xi(p, \theta), \theta) > 0$ holds.

(C 3) There exist measurable mappings $\hat{\theta}_n : R^n \to \Theta$ such that

$$\sqrt{n}\,(\,\hat{\theta}_n(x_1, \ldots, x_n) - \theta\,) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{l}(x_i, \theta) + o_P(1)\,, \tag{2.7}$$

where $o_P(1)$ is related to P_θ^∞ and

$$E_\theta(\,\mathbf{l}(x, \theta)\,) = \mathbf{0}\,, \qquad E_\theta(\,\|\mathbf{l}(x, \theta)\|^2)\ < +\infty\,. \tag{2.8}$$

(C 4) There exist an open interval U containing $\xi(p, \theta)$ and an open convex set V containing θ such that $f(t, \gamma)$ is continuous on $U \times V$.

(C 5) The function $\xi(p, \theta)$, defined in (2.3), has all partial derivatives

$$\frac{\partial \xi(p, \theta)}{\partial \theta_i}$$

of the first order, and these are continuous on Θ .

Theorem 2.1 *Let us assume that (2.5) holds, $\theta \in \Theta$, the conditions (C 1) - (C 5) are fulfilled ((C 2), (C 4) and (C 5) for $p = p_i$, $i = 1, \ldots, k$), $\{x_n\}_{n=1}^{\infty}$ are independent P_θ distributed random variables, $\{\hat{\theta}_n\}_{n=1}^{\infty}$ are the mappings from (C 3) and $\mathbf{\Delta_n}$ is the vector of differences determined with (2.1)-(2.6).*

(I) For $n \to \infty$ the weak convergence of distributions

$$\mathcal{L}(\sqrt{\mathbf{n}}\mathbf{\Delta_n}) \longrightarrow N(\mathbf{0}, \mathbf{\Sigma}(\theta)) \tag{2.9}$$

holds. Here the asymptotic covariance matrix

$$\mathbf{\Sigma}(\theta) = \mathbf{A} + \mathbf{D}(\theta)\mathbf{\Psi}(\theta)\mathbf{C}(\theta)' + \mathbf{C}(\theta)\mathbf{\Psi}(\theta)'\mathbf{D}(\theta) + \mathbf{D}(\theta)\mathbf{\Psi}(\theta)\mathbf{L}(\theta)\mathbf{\Psi}(\theta)'\mathbf{D}(\theta)\,, \tag{2.10}$$

$\mathbf{A}$ is the symmetric $k \times k$ matrix with the elements

$$a_{ij} = p_i(1 - p_j) \text{ for all } i \le j\,, \tag{2.11}$$

$\mathbf{D}(\theta)$ is the diagonal $k \times k$ matrix with the diagonal $f(\xi(p_1, \theta), \theta), \ldots, f(\xi(p_k, \theta), \theta)$,

$$\begin{aligned}
\mathbf{\Psi}(\theta) &= \begin{pmatrix} \dfrac{\partial \xi(p_1,\theta)}{\partial \theta_1} & , \; \ldots \; , & \dfrac{\partial \xi(p_1,\theta)}{\partial \theta_m} \\ \vdots & & \vdots \\ \dfrac{\partial \xi(p_k,\theta)}{\partial \theta_1} & , \; \ldots \; , & \dfrac{\partial \xi(p_k,\theta)}{\partial \theta_m} \end{pmatrix} \\
\mathbf{C}(\theta) &= \begin{pmatrix} cov\Big(\chi_{(-\infty, \xi(p_1,\theta)\rangle}(x) \,,\, \mathbf{l}(x,\theta) \Big) \\ \vdots \\ cov\Big(\chi_{(-\infty, \xi(p_k,\theta)\rangle}(x) \,,\, \mathbf{l}(x,\theta) \Big) \end{pmatrix},
\end{aligned} \tag{2.12}$$

$\mathbf{L}(\theta) = Var(\mathbf{l}(x, \theta))$ and all the covariances are related to P_θ.

(II) If the matrix (2.10) is regular, then for distribution of the statistics

$$T_n = n\mathbf{\Delta_n}'\mathbf{\Sigma}(\theta)^{-1}\mathbf{\Delta_n} \tag{2.13}$$

the weak convergence

$$\mathcal{L}(T_n) \longrightarrow \chi^2_k \tag{2.14}$$

holds as $n \to \infty$.

(III) Let $\mathbf{s}(x) = (s_1(x), \ldots, s_k(x))'$, $\mathbf{t}(x) = (t_1(x), \ldots, t_k(x))'$, where

$$s_i(x) = p_i - \chi_{(-\infty, \xi(p_i,\theta)\rangle}(x)\,, \qquad t_i(x) = f(\xi(p_i, \theta), \theta) \left(\frac{\partial \xi(p_i, \theta)}{\partial \theta} \right)' \mathbf{l}(x, \theta) \tag{2.15}$$

and $\mathbf{l}(., \theta)$ is the function from (C 3). The matrix (2.10) is regular if and only if

$$P_\theta(\,\mathbf{c}'\mathbf{s}(x) - \mathbf{c}'\mathbf{t}(x) = 0\,) < 1 \tag{2.16}$$

for all non-zero vectors $\mathbf{c}$ *from* R^k. *A sufficient condition for this is that for all* $i = 1, \dots, k+1$, *each non-zero vector* $\mathbf{a} \in R^k$ *and every non-zero real number* d

$$P_\theta(\,\{x \in (\xi(p_{i-1}, \theta), \xi(p_i, \theta));\; \mathbf{a}'\mathbf{l}(x, \theta) \neq d\,\}\,) > 0\,. \tag{2.17}$$

Here we use the notation

$$p_0 = 0\,,\quad \xi(0, \theta) = -\infty\,,\qquad p_{k+1} = 1\,,\quad \xi(1, \theta) = +\infty\,. \tag{2.18}$$

Since according to (2.9), (2.7) and (2.8)

$$\Delta_n = \mathcal{O}_P(n^{-1/2})\,,\qquad \hat{\theta}_n = \theta + o_P(1)\,,$$

from the previous theorem we obtain immediately the following assertion.

Corollary *Let the assumptions of Theorem 2.1 hold for each* $\theta \in \Theta$. *If the matrix function (2.10) is continuous on* Θ *and takes its values in the set of the regular* $k \times k$ *matrices, then with the notation from the previous theorem and*

$$T_n = n\boldsymbol{\Delta}_n' \boldsymbol{\Sigma}(\hat{\theta}_n)^{-1} \boldsymbol{\Delta}_n \tag{2.19}$$

the weak convergence of distributions

$$\mathcal{L}(T_n | P_\theta) \longrightarrow \chi_k^2 \tag{2.20}$$

holds for each $\theta \in \Theta$ *as* $n \to \infty$.

Under the assumptions of the previous corollary the null hypothesis that the true distribution P belongs to the family $\{P_\theta;\ \theta \in \Theta\}$ is rejected at the asymptotic significance level α, if (2.19) exceeds the critical value $\chi_k^2(\alpha)$ of the chi-square distribution with k degrees of freedom. In the next part of this section we pay attention to a situation when the asymptotic covariance matrix does not depend on the true value of the parameter.

Let us assume that

$$f : R \to \langle 0, +\infty)\,,\qquad \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1\,, \tag{2.21}$$

and put

$$F(t) = \int_{-\infty}^{t} f(x)\,\mathrm{d}x\,, \tag{2.22}$$

$$\Theta = \{\,\begin{pmatrix} \mu \\ \sigma \end{pmatrix} \in R^2;\ \sigma > 0\,\}\,. \tag{2.23}$$

For $\theta = (\mu, \sigma)' \in \Theta$ let

$$f(x, \theta) = \frac{1}{\sigma} f\left(\frac{x - \mu}{\sigma} \right), \qquad F(t, \theta) = F\left(\frac{t - \mu}{\sigma} \right) \tag{2.24}$$

denote density and distribution function of the probability P_θ. One can easily find out that for the quantiles (2.3) the equalities

$$\xi(p, \theta) = \sigma \xi_p + \mu \,, \qquad \xi_p = \xi(p, (0,1)') \tag{2.25}$$

hold.

Theorem 2.2 *Let us assume that in the setting (2.21) - (2.24) the measurable mappings $\hat{\theta}_n : R^n \to \Theta$ are such that for every real $a > 0$, b and every $\theta = (\mu, \sigma)' \in \Theta$*

$$\hat{\mu}_n(az_1 + b, \ldots, az_n + b) \;=\; a\hat{\mu}_n(z_1, \ldots, z_n) + b \,, \tag{2.26}$$

$$\hat{\sigma}_n(az_1 + b, \ldots, az_n + b) \;=\; a\hat{\sigma}_n(z_1, \ldots, z_n) \tag{2.27}$$

a. e. P_θ. Let us further assume that $P_{\theta_0}(\,\hat{\sigma}_n > 0\,) = 1$ for some $\theta_0 \in \Theta$ and all $n > n_0$.

(I) The vector $\boldsymbol{\Delta}_n$ of differences (2.6) has an exact null distribution, i.e., the distribution

$$\mathcal{L}\Big(\boldsymbol{\Delta}_\mathbf{n} \,|\, P_\theta\Big) = \mathcal{L}\Big(\boldsymbol{\Delta}_\mathbf{n} \,|\, P_{(0,1)'}\Big) \tag{2.28}$$

does not depend on $\theta \in \Theta$ for all $n > n_0$.

(II) Let the numbers (2.5) be fixed. If (C 1) - (C 4) hold for $\mu = 0$, $\sigma = 1$ and $p_1, \ldots, p_k$, then the assumptions of Theorem 2.1(I) are valid for all $\theta \in \Theta$ and the limit covariance matrix (2.10) does not depend on θ. If this matrix $\boldsymbol{\Sigma}$ is regular, then for distributions of statistics

$$T_n = n \boldsymbol{\Delta}_n' \boldsymbol{\Sigma}^{-1} \boldsymbol{\Delta}_n \tag{2.29}$$

the weak convergence

$$\mathcal{L}(T_n | P_\theta) \longrightarrow \chi^2_k \tag{2.30}$$

holds for each $\theta \in \Theta$ as $n \to \infty$.

Now we are going to discuss briefly the assumption of regularity of the asymptotic covariance matrix in the assertion (II) of the previous theorem. Let in the setting (2.21) - (2.24) the conditions (A1) - (A3) and (A6) from [12] hold for all $\theta \in \Theta$. Then for all real x the derivative $f'(x)$ exists and is continuous on the real line. If in addition to this f is positive on the interval (d, D), where

$$d = \inf\{x;\; f(x) > 0\}\,, \qquad D = \sup\{x;\; f(x) > 0\}\,,$$

then for all $\theta \in \Theta$ the conditions (C 1) - (C 5) of this paper hold and one can prove by means of the condition related to (2.16) that the matrix (2.10) is regular. Since assumptions of this paper are less stringent than the mentioned assumptions from [12], the situation is in this general case of the location and scale parameter more complicated. For this reason we apply Theorem 2.2 to the two parameter exponential distribution and to the Laplace distribution in such a way that we prove regularity of the asymptotic covariance matrix by verifying the simple conditions presented in the assertion (III) of Theorem 2.1.

Let us assume that $\{P_\theta;\, \theta \in \Theta\}$ is the family of the two parameter exponential distributions, determined with the parameter set (2.23) and the densities

$$f(x,\mu,\sigma) = \begin{cases} \frac{1}{\sigma}\exp\left[-\frac{(x-\mu)}{\sigma}\right] & x \geq \mu\,, \\ 0 & x < \mu\,, \end{cases} \tag{2.31}$$

with respect to the Lebesque measure on the real line. Thus in the notation (2.21) - (2.24)

$$f(x) = \begin{cases} \mathrm{e}^{-x} & x \geq 0\,, \\ 0 & x < 0\,, \end{cases} \qquad F(t) = \begin{cases} 1-\mathrm{e}^{-t} & t \geq 0\,, \\ 0 & t < 0\,. \end{cases} \tag{2.32}$$

Obviously, the quantiles (2.3) fulfil the equalities

$$\xi(p,\mu,\sigma) = \sigma\xi_p + \mu\,, \qquad \xi_p = -\log(1-p)\,. \tag{2.33}$$

Exponential distribution with density (2.31) will be denoted by the symbol $E(\mu,\sigma)$. In the following theorem we base the quantile test statistic on the maximum likelihood estimate (2.34).

Theorem 2.3 *Let us assume that the vector of differences (2.6) is determined with (2.24), (2.31) - (2.34) and that* $\hat{\theta}_n = \hat{\theta}_n(x_1,\ldots,x_n) = (\hat{\mu},s)'$, *where*

$$\hat{\mu} = \min\{x_1,\ldots,x_n\}\,, \qquad s = \overline{x} - \hat{\mu}\,, \quad \overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j\,. \tag{2.34}$$

(I) The random vector $\mathbf{\Delta_n}$ *has an exact null distribution, i.e.,*

$$\mathcal{L}\Big(\mathbf{\Delta_n} \,|\, E(\mu,\sigma)\Big) = \mathcal{L}\Big(\mathbf{\Delta_n} \,|\, E(0,1)\Big) \tag{2.35}$$

for all $n > 1$ *and* $(\mu,\sigma)' \in \Theta$.

(II) As $n \to \infty$,

$$\mathcal{L}\Big(\sqrt{\mathbf{n}}\mathbf{\Delta_n} \,|\, E(0,1)\Big) \longrightarrow N(\mathbf{0},\mathbf{\Sigma}) \tag{2.36}$$

in the sense of the weak convergence of probability measures. Here

$$\mathbf{\Sigma} = \mathbf{A} - \mathbf{b}\mathbf{b}'\,, \tag{2.37}$$

$\mathbf{A}$ *is the symmetric* $k \times k$ *matrix with the elements (2.11) and the vector* $\mathbf{b} = (\beta_1,\ldots,\beta_k)'$ *has the coordinates*

$$\beta_i = -(1-p_i)\log(1-p_i)\,. \tag{2.38}$$

(III) Let us put

$$\alpha_i = p_i - p_{i-1}\,, \tag{2.39}$$

$$p_0 = 0\,, \qquad p_{k+1} = 1\,, \tag{2.40}$$

and $\beta_0 = \beta_{k+1} = 0$. Then

$$\sum_{i=1}^{k+1} \frac{(\beta_i - \beta_{i-1})^2}{\alpha_i} < 1 \tag{2.41}$$

and the matrix (2.37) is regular. The statistic (2.29) can be written in the form

$$T_n = n\mathbf{\Delta_n}'\mathbf{\Sigma}^{-1}\mathbf{\Delta_n} =$$

$$= n\sum_{i=1}^{k+1} \frac{\Big(P_{\hat\theta_n}(C_i^{(n)}) - \alpha_i\Big)^2}{\alpha_i} + n\,\frac{\left(\displaystyle\sum_{i=1}^{k+1} \frac{(\beta_i - \beta_{i-1})\Big(P_{\hat\theta_n}(C_i^{(n)}) - \alpha_i\Big)}{\alpha_i}\right)^2}{1 - \displaystyle\sum_{i=1}^{k+1} \frac{(\beta_i - \beta_{i-1})^2}{\alpha_i}}\,, \tag{2.42}$$

where (cf. (2.2), (2.24) and (2.32))

$$C_i^{(n)} = (\hat\xi_{p_{i-1},n}\,,\,\hat\xi_{p_i,n}\rangle\,, \qquad P_{\hat\theta_n}(C_i^{(n)}) = F(\hat\xi_{p_i,n}\,,\hat\theta_n) - F(\hat\xi_{p_{i-1},n}\,,\hat\theta_n)\,, \tag{2.43}$$

$$\hat\xi_{0,n} = -\infty\,, \qquad \hat\xi_{1,n} = +\infty\,. \tag{2.44}$$

(IV) For every $(\mu,\sigma)' \in \Theta$ and the statistics (2.42)

$$\mathcal{L}(\,T_n \,|\, E(\mu,\sigma)\,) \longrightarrow \chi^2_k \tag{2.45}$$

as $n \to \infty$.

Remark. Let us assume that $\Theta = (0,+\infty)$ and $\{\,f(x,\theta);\ \theta \in \Theta\,\}$ are the densities

$$f(x,\theta) = \frac{1}{\theta}\exp\left[-\frac{x}{\theta}\right]\,\chi_{\langle 0,+\infty)}(x)$$

of the one parameter exponential distributions $E(\theta)$. For random samples $x_1,\ldots,x_n$ from such a distribution let

$$\hat\theta_n(x_1,\ldots,x_n) = \frac{1}{n}\sum_{j=1}^{n} x_j\,.$$

Then distribution function of the exponential $E(\hat\theta_n)$ distribution $F(t,\hat\theta_n) = F(\frac{t}{\hat\theta_n})$, where F is the function (2.32). Proceeding similarly as in the proof of the previous theorem one easily finds out that for the differences (2.6)

$$\mathcal{L}\Big(\mathbf{\Delta_n} \,|\, E(\theta)\Big) = \mathcal{L}\Big(\mathbf{\Delta_n} \,|\, E(1)\Big)$$

for every $\theta > 0$ and $n > 1$, the assumptions used in Theorem 2.1 are fulfilled and the statistics (2.13) equal (2.42), where now with the notation from (2.32)

$$P_{\hat{\theta}_n}(C_i^{(n)}) = F\left(\frac{\hat{\xi}_{p_i,n}}{\hat{\theta}_n}\right) - F\left(\frac{\hat{\xi}_{p_{i-1},n}}{\hat{\theta}_n}\right).$$

Also $\mathcal{L}(\,T_n \mid E(\theta)\,) \longrightarrow \chi_k^2$ as $n \to \infty$.

Since the density (2.31) is discontinuous at $x = \mu$ and in the case (3.15) the condition *(C 3)* holds with (3.17) which has a singular covariance matrix, the classical conditions from [6], [12], [13], [14], [19] and [20] are not fulfilled by the exponential distributions. A similar situation occurs for the Laplace distribution treated in the further text.

The Pareto distribution $P(k,a)$ is in [9], p.574 defined as the distribution having the density $f(z,k,a) = ak^a z^{-(a+1)}\chi_{\langle k,+\infty)}(z)$, where the parameters k, a are positive real numbers. However, according to (20.9) on p. 576 in [9] a random variable z has the $P(k,a)$ distribution if and only if $x = \log z$ has the exponential $E(\mu,\sigma)$ distribution, where $\sigma = a^{-1}$, $\mu = \log k$ and log denotes the logarithm to the base e. Thus making use of the transformation

$$x = \log z$$

we can utilize the results from the previous theorem also for testing the null hypothesis that the random variable z has a Pareto distribution.

In the last part of this section we assume that $\{P_\theta;\ \theta \in \Theta\}$ is the family of the Laplace distributions, determined with the parameter set (2.23) and the densities

$$f(x,\mu,\sigma) = \frac{1}{2\sigma}\exp\left[-\frac{|x-\mu|}{\sigma}\right] \tag{2.46}$$

with respect to the Lebesque measure on the real line. Thus in the notation (2.21) - (2.24)

$$f(x) = \frac{e^{-|x|}}{2}, \qquad F(t) = \begin{cases} 1 - \dfrac{e^{-t}}{2} & t \geq 0\,, \\[2ex] \dfrac{e^{t}}{2} & t < 0\,. \end{cases} \tag{2.47}$$

The quantiles (2.3) fulfil the relations

$$\xi(p,\mu,\sigma) = \sigma\xi_p + \mu\,, \qquad \xi_p = \begin{cases} \log 2p & p \in (0, \frac{1}{2}\rangle\,, \\[2ex] -\log 2(1-p) & p \in \langle \frac{1}{2}, 1)\,. \end{cases} \tag{2.48}$$

In the following theorem $L(\mu,\sigma)$ denotes the Laplace distribution determined with density (2.46). The quantile test statistic is based on the maximum likelihood estimate (2.49), where $x^{(j)} = x_n^{(j)}$ denotes the j-th order statistic.

Theorem 2.4 *Let us assume that the vector of differences (2.6) is determined with (2.24), (2.46) - (2.49) and that* $\hat{\theta}_n = \hat{\theta}_n(x_1, \ldots, x_n) = (\hat{\mu}, s)'$*, where*

$$\hat{\mu} = \begin{cases} \dfrac{x^{(k)} + x^{(k+1)}}{2} & n = 2k\,, \\[2ex] x^{(k+1)} & n = 2k+1\,, \end{cases} \qquad s = \frac{1}{n}\sum_{j=1}^{n} |x_j - \hat{\mu}|\,. \tag{2.49}$$

(I) The random vector $\mathbf{\Delta_n}$ *has an exact null distribution, i.e.,*

$$\mathcal{L}\Big(\mathbf{\Delta_n} \,|\, L(\mu,\sigma)\Big) = \mathcal{L}\Big(\mathbf{\Delta_n} \,|\, L(0,1)\Big) \tag{2.50}$$

for all $n > 1$ *and* $(\mu, \sigma)' \in \Theta$.

(II) As $n \to \infty$,

$$\mathcal{L}\Big(\sqrt{\mathbf{n}}\mathbf{\Delta_n} \,|\, L(0,1)\Big) \longrightarrow N(\mathbf{0}, \mathbf{\Sigma}) \tag{2.51}$$

in the sense of the weak convergence of probability measures. Here

$$\mathbf{\Sigma} = \mathbf{A} - \mathbf{B}\mathbf{B}'\,, \tag{2.52}$$

$\mathbf{A}$ *is the symmetric* $k \times k$ *matrix determined with (2.11) and*

$$\mathbf{B} = \begin{pmatrix} f(\xi p_1) & , & \xi_{p_1} f(\xi p_1) \\ & \vdots & \\ f(\xi p_k) & , & \xi_{p_k} f(\xi p_k) \end{pmatrix}. \tag{2.53}$$

(III) The covariance matrix (2.52) is regular if and only if $\frac{1}{2} \notin \{p_1, \ldots, p_k\}$*. In this case also the matrix* $\mathbf{I_2} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}$ *is regular and the statistic (2.29) can be written in the form (cf. (2.39), (2.40), (2.43) and (2.44))*

$$T_n = n\mathbf{\Delta_n}'\mathbf{\Sigma}^{-1}\mathbf{\Delta_n} = n\sum_{i=1}^{k+1} \frac{\Big(P_{\hat{\theta}_n}(C_i^{(n)}) - \alpha_i\Big)^2}{\alpha_i} + nQ(\mathbf{\Delta_n})\,, \tag{2.54}$$

where

$$Q(\mathbf{\Delta_n}) = \mathbf{\Delta_n}'\mathbf{A}^{-1}\mathbf{B}(\mathbf{I_2} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{A}^{-1}\mathbf{\Delta_n}\,, \tag{2.55}$$

and for every $(\mu,\sigma)' \in \Theta$

$$\mathcal{L}(\,T_n \,|\, L(\mu,\sigma)\,) \longrightarrow \chi^2_k \tag{2.56}$$

as $n \to \infty$.

The formula (2.55) can be expressed in a more detailed way. Indeed, putting

$$r_{x,y} = \frac{x_1 y_1}{\alpha_1} + \sum_{i=2}^{k} \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{\alpha_i} + \frac{x_k y_k}{\alpha_{k+1}}\,, \tag{2.57}$$

$$\lambda_{11} = 1 - r_{f,f}\,, \qquad \lambda_{12} = r_{f,\xi f}\,, \qquad \lambda_{22} = 1 - r_{\xi f,\xi f}\,, \tag{2.58}$$

$$f \;=\; (\,f(\xi_{p_1}),\ldots,f(\xi_{p_k})\,)'\;, \qquad \xi f = (\xi_{p_1} f(\xi_{p_1}),\ldots,\xi_{p_k} f(\xi_{p_k}))' \tag{2.59}$$

and applying (3.13) to (2.55) we see that

$$Q(\mathbf{\Delta_n}) = \frac{(r_{\mathbf{\Delta_n},f})^2\lambda_{22} + 2\,r_{\mathbf{\Delta_n},f}\,r_{\mathbf{\Delta_n},\xi f}\lambda_{12} + (r_{\mathbf{\Delta_n},\xi f})^2\lambda_{11}}{\lambda_{11}\lambda_{22} - \lambda_{12}^2}\;. \tag{2.60}$$

If $p_i = \frac{1}{2}$ for some i and the sampling is made from the Laplace distribution, then according to the previous theorem the statistic (2.54) cannot be constructed because of singularity of the asymptotic covariance matrix. However, this difficulty can be overcome when instead of (2.49) another (even though less efficient) estimate is used.

Theorem 2.5 *Let us assume that the vector of differences (2.6) is determined with (2.24), (2.46) – (2.47) and that* $\hat\theta_n = \hat\theta_n(x_1,\ldots,x_n) = (\overline{x}, s)'$*, where* s *is defined by the formula (2.49) and* $\overline{x}$ *is the arithmetic mean.*

(I) The random vector $\mathbf{\Delta_n}$ *has an exact null distribution, i.e.,*

$$\mathcal{L}\Big(\mathbf{\Delta_n}\,|\,L(\mu,\sigma)\Big) = \mathcal{L}\Big(\mathbf{\Delta_n}\,|\,L(0,1)\Big) \tag{2.61}$$

for all $n > 1$ *and* $(\mu,\sigma)' \in \Theta$.

(II) As $n \to \infty$,

$$\mathcal{L}\Big(\sqrt{\mathbf{n}}\mathbf{\Delta_n}\,|\,L(0,1)\Big) \longrightarrow N(\mathbf{0},\mathbf{\Sigma}) \tag{2.62}$$

in the sense of the weak convergence of probability measures. Here

$$\mathbf{\Sigma} = \mathbf{A} - \mathbf{G}\,, \tag{2.63}$$

$\mathbf{A}$ *is the symmetric* $k \times k$ *matrix determined with (2.11) and* $\mathbf{G}$ *is the symmetric* $k \times k$ *matrix whose elements are defined by the formula (cf. (2.47), (2.48))*

$$g_{ij} = f(\xi_{p_i})f(\xi_{p_j})\Big(|\xi_{p_i}| + |\xi_{p_j}| + \xi_{p_i}\xi_{p_j}\Big)\,. \tag{2.64}$$

The matrix (2.63) is regular and for $T_n = n\mathbf{\Delta_n}'\mathbf{\Sigma}^{-1}\mathbf{\Delta_n}$ *the weak convergence of distributions*

$$\mathcal{L}(\,T_n\,|\,L(\mu,\sigma)\,) \longrightarrow \chi^2_k \tag{2.65}$$

holds for each $\theta \in \Theta$ *as* $n \to \infty$.

3. PROOFS

P r o o f o f T h e o r e m 2. 1. (I) Let $p \in \{p_1,\ldots,p_k\}$ be fixed. Since *(C 4)* and *(C 2)* hold, according to Theorem 1 in [7] in the setting (2.1) – (2.4)

$$\hat\xi_{p,n} - \xi(p,\theta) = \frac{p - \hat F_n(\xi(p,\theta))}{f(\xi(p,\theta)\,,\,\theta)} \;+\; o_P(n^{-1/2})\,. \tag{3.1}$$

Validity of *(C 5)*, *(C 3)* and the central limit theorem imply that

$$\xi(p,\hat{\theta}_n)-\xi(p,\theta)=\left(\frac{\partial\xi(p,\theta)}{\partial\theta}\right)'(\hat{\theta}_n-\theta)\;+\;o_P(n^{-1/2})\,. \tag{3.2}$$

Hence with probability tending to 1 as $n\to\infty$, the relations

$$F(\hat{\xi}_{p,n},\hat{\theta}_n)-p\;=\;F(\hat{\xi}_{p,n},\hat{\theta}_n)-F(\xi(p,\hat{\theta}_n),\hat{\theta}_n)\;=$$

$$=\;\frac{\partial F(t,\hat{\theta}_n)}{\partial t}\Big|_{t=\alpha\hat{\xi}_{p,n}+(1-\alpha)\xi(p,\hat{\theta}_n)}(\hat{\xi}_{p,n}-\xi(p,\hat{\theta}_n))=$$

$$=\;f\Big(\xi(p,\theta),\theta\Big)\,(\,\hat{\xi}_{p,n}-\xi(p,\hat{\theta}_n)\,)+\beta_n \tag{3.3}$$

hold with

$$\beta_n=\Big(f\Big(\,\alpha\hat{\xi}_{p,n}+(1-\alpha)\xi(p,\hat{\theta}_n\,),\hat{\theta}_n\,\Big)-f\Big(\,\xi(p,\theta),\theta\,\Big)\Big)\,\Big(\,\hat{\xi}_{p,n}-\xi(p,\hat{\theta}_n)\Big)\,.$$

Owing to (3.1), (3.2), *(C 4)*, *(C 3)* and the central limit theorem

$$|\beta_n|\leq o_P(1)\mathcal{O}_P(n^{-1/2})=o_P(n^{-1/2})\,. \tag{3.4}$$

Combining (3.4), (3.3), (3.1) and (3.2) we get the relation

$$F(\hat{\xi}_{p,n},\hat{\theta}_n)-p=p-\hat{F}_n(\xi(p,\theta))-f(\xi(p,\theta),\theta)\left(\frac{\partial\xi(p,\theta)}{\partial\theta}\right)'(\hat{\theta}_n-\theta)+o_P(n^{-1/2})\,. \tag{3.5}$$

It follows from this, (2.6) and (2.7) that

$$\sqrt{n}\,\mathbf{\Delta_n}=\mathbf{M}\sqrt{n}\,\eta_n+o_P(1)\,, \tag{3.6}$$

where with the notation from wording of the theorem

$$\mathbf{M}=(\mathbf{I_k},\mathbf{D\Psi})\,,\qquad\quad \eta_n=\begin{pmatrix} p_1-\hat{F}_n(\,\xi(p_1,\theta)\,)\\ \\ \vdots\\ \\ p_k-\hat{F}_n(\,\xi(p_k,\theta)\,)\\ \\ -\frac{1}{n}\sum_{i=1}^n\mathbf{l}(x_i,\theta)\end{pmatrix}.$$

Since according to the central limit theorem

$$\mathcal{L}(\sqrt{n}\,\eta_n)\to N(\mathbf{0},\mathbf{V})\,,\qquad \mathbf{V}=\begin{pmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C}' & \mathbf{L}\end{pmatrix},$$

the assertion (I) follows from (3.6).

(II) The proof follows immediately from (2.9).

(III) The proof of the first part of the assertion follows from the fact that according to (3.5) and *(C 3)*

$$\mathbf{\Sigma}(\theta) = \operatorname{cov}(\mathbf{s}(x) - \mathbf{t}(x)), \qquad E_\theta(\mathbf{s}(x) - \mathbf{t}(x)) = \mathbf{0}.$$

In proving the implication from (2.17) to (2.16) we assume that (2.17) holds in the sense of the assertion of the theorem, choose a vector $\mathbf{c} \in R^k$ and suppose that

$$P_\theta(\mathbf{c}'\mathbf{s}(x) - \mathbf{c}'\mathbf{t}(x) = 0) = 1, \tag{3.7}$$

or equivalently, that

$$P_\theta(\mathbf{a}'\mathbf{l}(x,\theta) = \mathbf{c}'\mathbf{s}(x)) = 1, \tag{3.8}$$

where

$$\mathbf{a} = \sum_{i=1}^{k} c_i f(\,\xi(p_i,\theta),\theta\,)\,\frac{\partial \xi(p_i,\theta)}{\partial\theta}.$$

Suppose that $\mathbf{a} \neq \mathbf{0}$. Then $\mathbf{c} \neq \mathbf{0}$ and after some computation we get from (3.8) that $\mathbf{a}'\mathbf{l}(x,\theta)$ differs from a non-zero constant on some of the intervals $(\,\xi(p_{i-1},\theta), \xi(p_i,\theta)\,)$ with probability P_θ equal to zero, which is a contradiction with our assumption. Thus

$$\mathbf{a} = \mathbf{0}$$

and $\mathbf{c}'\mathbf{s}(x) = 0$ a.e. P_θ. Hence for $x \in (\,\xi(p_k,\theta), +\infty\,)$ we obtain that

$$\sum_{i=1}^{k} c_i p_i = 0 \tag{3.9}$$

and for $x \in (\,\xi(p_{j-1},\theta), \xi(p_j,\theta)\,)$

$$\sum_{i=1}^{j-1} c_i p_i + \sum_{i=j}^{k} c_i(p_i - 1) = 0, \qquad j = 1,\dots,k. \tag{3.10}$$

From (3.9) and (3.10) we easily get that $c_1 = \ldots = c_k = 0$, and (2.16) holds for all non-zero vectors $\mathbf{c} \in R^k$. $\square$

Proof of Theorem 2. 2. (I) For real numbers $z_1,\dots,z_n$ and $0 < p < 1$ let

$$F_n^z(t) = \frac{1}{n}\sum_{j=1}^{n} \chi_{(-\infty,t)}(z_j), \qquad \hat{z}_{p,n} = \inf\{t;\, F_n^z(t) \geq p\}.$$

If $\sigma > 0$, μ are real numbers and for $x_j = \sigma z_j + \mu$ the quantities $F_n^x(t)$, $\hat{x}_{p,n}$ are defined in the same way, then

$$F_n^x(t) = F_n^z(\frac{t-\mu}{\sigma}), \qquad \hat{x}_{p,n} = \sigma\hat{z}_{p,n} + \mu, \tag{3.11}$$

which together with (2.24) - (2.27) implies (2.28).

(II) Let *(C 1) - (C 4)* hold for

$$\vartheta = (0, 1)' . \tag{3.12}$$

Putting for $\theta = (\mu, \sigma)' \in \Theta$

$$\mathbf{l}(x, \theta) = \sigma \mathbf{l}(\frac{x - \mu}{\sigma}, \vartheta)$$

and utilizing (2.24) - (2.27) and (3.11), one can easily verify validity of *(C 1) - (C 5)* for θ. The equality $\mathbf{\Sigma}(\theta) = \mathbf{\Sigma}(\vartheta)$ follows from (I) and (2.30) is obvious. □

In deriving explicit formulas for the quantile test statistics the following assertion will be useful.

Lemma 3.1 *The symmetric $k \times k$ matrix $\mathbf{A}$ having the elements (2.11) is regular, and for every vectors x , $y \in R^k$ (cf. (2.39), (2.40))*

$$x' \mathbf{A}^{-1} y = \frac{x_1 y_1}{\alpha_1} + \sum_{i=2}^{k} \frac{(x_i - x_{i-1})(y_i - y_{i-1})}{\alpha_i} + \frac{x_k y_k}{\alpha_{k+1}} . \tag{3.13}$$

Proof. Let $\mathbf{B} = \left(b_{ij}\right)$ be the $k \times k$ matrix with $b_{ij} = 1$ if $j \le i$, and $b_{ij} = 0$ otherwise. If $\mathbf{D}$ denotes diagonal matrix with the diagonal $\alpha_1, \dots, \alpha_k$ and if $\alpha' = (\alpha_1, \dots, \alpha_k)$, then

$$\mathbf{A} = \mathbf{B}\mathbf{G}\mathbf{B}' , \quad \mathbf{G} = \mathbf{D} - \alpha\, \alpha' .$$

Since $\alpha' \mathbf{D}^{-1} \alpha = p_k < 1$, according to the excercise 2.8, section 1.b in [16]

$$(\mathbf{D} - \alpha\, \alpha')^{-1} = \mathbf{D}^{-1} + \frac{\mathbf{D}^{-1} \alpha \alpha' \mathbf{D}^{-1}}{1 - \alpha' \mathbf{D}^{-1} \alpha} .$$

Thus

$$\mathbf{A}^{-1} = (\mathbf{B}^{-1})' \left[\mathbf{D}^{-1} + \frac{1}{\alpha_{k+1}} \mathbf{1}\mathbf{1}' \right] \mathbf{B}^{-1} \tag{3.14}$$

where $\mathbf{1}$ is the vector from R^k having all coordinates 1, and $\mathbf{B}^{-1} = \left(b^{ij}\right)$ with $b^{ij} = 1$ for $j = i$, $b^{ij} = -1$ for $j = i - 1$ and $b^{ij} = 0$ otherwise. Obviously, (3.14) implies assertion of the lemma. □

Proof of Theorem 2.3. (I) This assertion follows from Theorem 2.2(I).

(II) Let us assume that

$$\theta = \vartheta \tag{3.15}$$

where ϑ is the parameter (3.12). Then

$$\hat{\mu} = o_P(n^{-1/2}) \tag{3.16}$$

and the relations (2.7), (2.8) hold with

$$\mathbf{l}(x,\vartheta) = \begin{pmatrix} 0 \\ x-1 \end{pmatrix} . \tag{3.17}$$

Thus it is obvious that the conditions *(C 1) - (C 4)* are valid, which according to Theorem 2.2(II) means that the assumptions of Theorem 2.1(I) are fulfilled and (2.36) holds. Since in (2.10) for $\theta = \vartheta$

$$\mathbf{\Psi} = \begin{pmatrix} 1 \ , \ -\log(1-p_1) \\ \vdots \\ 1 \ , \ -\log(1-p_k) \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 0 \ , \ (1-p_1)\log(1-p_1) \\ \vdots \\ 0 \ , \ (1-p_k)\log(1-p_k) \end{pmatrix}, \ \mathbf{L} = \begin{pmatrix} 0 \ , \ 0 \\ \\ 0 \ , \ 1 \end{pmatrix}$$

and the matrix $\mathbf{D}$ has the diagonal $1-p_1,\ldots,1-p_k$, the matrix (2.10) equals (2.37).

(III) If $\mathbf{a} \in R^2$ is a non-zero vector, then for the function (3.17) the equality $\mathbf{a}'\mathbf{l}(x,\vartheta) = a_2(x-1)$ holds and regularity of $\mathbf{\Sigma}$ can be easily proved by means of the condition from Theorem 2.1 related to (2.17). Employing results of the excercise 2.4, section 1.b in [16], we see that

$$0 < |\mathbf{\Sigma}| \ = |\mathbf{A} - \mathbf{b}\,\mathbf{b}'| = \begin{vmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}' & 1 \end{vmatrix} = |\mathbf{A}|(1 - \mathbf{b}'\mathbf{A}^{-1}\mathbf{b}) \tag{3.18}$$

and therefore

$$1 \ > \ \mathbf{b}'\mathbf{A}^{-1}\mathbf{b} \ = \ \sum_{i=1}^{k+1} \frac{(\beta_i - \beta_{i-1})^2}{\alpha_i} \tag{3.19}$$

where the last equality follows from (3.13). Validity of (3.19) together with excercise 2.8, section 1.b in [16] lead to the equality

$$\mathbf{\Sigma}^{-1} = \mathbf{A}^{-1} \ + \ \frac{\mathbf{A}^{-1}\mathbf{b}\mathbf{b}'\mathbf{A}^{-1}}{1 - \mathbf{b}'\mathbf{A}^{-1}\mathbf{b}}$$

and the assertion can be easily proved by means of (3.13).

(IV) This assertion follows from (I) - (III). □

P r o o f o f T h e o r e m 2. 4. (I) This assertion follows from Theorem 2.2(I).

(II) Suppose that (3.12), (3.15) hold. Since $x_1,\ldots,x_n$ are independent $L(0,1)$ distributed random variables, for the estimates (2.49) the equalities

$$\hat{\mu} \ = \ \frac{1}{n}\sum_{j=1}^{n} \mathrm{sign}(x_j) + o_P(n^{-1/2}) \, , \tag{3.20}$$

$$s \ = \ \frac{1}{n}\sum_{j=1}^{n} |x_j| + o_P(n^{-1/2}) \tag{3.21}$$

hold. Their validity can be established in various ways, perhaps the most convenient is the use of results from [8] and [15]. Indeed, (3.20) follows from Corollary 3.2 on p. 133 of [8]. Validity of (3.21) can be established by employing Example 5.3 on pp. 351 – 352 of [15] and carrying out the steps described on p. 342 in [15] with (2) ibidem as the final step tool.

Since $E(\text{sign}(x)) = 0$, $E(|x|) = 1$ and $E(x^2) = 2$, we obtain that the estimate (2.49) fufils the condition *(C 3)* with

$$\mathbf{l}(x,\vartheta) = \begin{pmatrix} \text{sign}(x) \\ |x|-1 \end{pmatrix} . \tag{3.22}$$

Since validity of *(C 4)* is obvious, we see that the conditions *(C 1)* - *(C 4)* are valid. This according to Theorem 2.2(II) means that the assumptions of Theorem 2.1(I) are fulfilled and (2.51) holds.

Utilizing (2.47) and (2.48) one can prove after some computation that for all $p \in (0,1)$

$$\begin{aligned} cov(\chi_{(-\infty,\xi_p\rangle}(x), \text{sign}(x)) &= -f(\xi_p)\,, \\ cov(\chi_{(-\infty,\xi_p\rangle}(x), |x|-1) &= -\xi_p f(\xi_p)\,. \end{aligned} \tag{3.23}$$

Thus in (2.10) for $\theta = \vartheta$

$$\mathbf{\Psi} = \begin{pmatrix} 1\ ,\ \xi_{p_1} \\ \vdots \\ 1\ ,\ \xi_{p_k} \end{pmatrix}, \quad \mathbf{C} = -\begin{pmatrix} f(\xi_{p_1})\ ,\ \xi_{p_1} f(\xi_{p_1}) \\ \vdots \\ f(\xi_{p_k})\ ,\ \xi_{p_k} f(\xi_{p_k}) \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 1\ ,\ 0 \\ \\ 0\ ,\ 1 \end{pmatrix} \tag{3.24}$$

and the matrix (2.10) equals (2.52).

(III) We shall verify regularity of the matrix (2.52) by means of the condition related to (2.16).

If $p_i = \frac{1}{2}$, then taking into account (2.15), (3.22) we see that $s_i(x) = t_i(x)$ a.e. P_ϑ, and according to Theorem 2.1(III) the asymptotic covariance matrix is singular.

Now let $p_i \neq \frac{1}{2}$ for all i. Let us assume that $\mathbf{c} \in R^k$ and the equality

$$\mathbf{c}'\mathbf{s}(x) - \mathbf{c}'\mathbf{t}(x) = 0 \tag{3.25}$$

holds almost everywhere $L(0,1)$. Letting x tend to infinity we get from (3.25) that

$$\sum_{i=1}^{k} c_i \xi_{p_i} f(\xi_{p_i}) = 0$$

and the equality

$$\sum_{i=1}^{k} c_i \chi_{(-\infty,\xi_{p_i}\rangle}(x) + \sum_{i=1}^{k} c_i f(\xi_{p_i}) \text{sign}(x) = \sum_{i=1}^{k} c_i p_i \tag{3.26}$$

holds for all $x \notin \{\xi_{p_1}, \ldots, \xi_{p_k}\}$.

If $\xi_{p_k} < 0$, then the interval $(\xi_{p_k}, +\infty)$ contains both positive and negative numbers. Hence the second and the third term in (3.26) are zero and (3.26) obviously implies that $c_1 = \ldots = c_k = 0$.

Since $\frac{1}{2} \neq p_i$ for all i, it remains to consider the case $\xi_{p_k} > 0$. If $\xi_{p_1} > 0$, then similarly as in the previous step $c_1 = \ldots = c_k = 0$. If $\xi_{p_1} < 0$, then denoting $j_0 = \min\{j;\, \xi_{p_j} > 0\}$ and choosing x from the particular intervals we obtain from (3.26) that

$$\begin{aligned} \sum_{i=j}^{k} c_i + \sum_{i=1}^{k} c_i f(\xi_{p_i}) &= \sum_{i=1}^{k} c_i p_i\,, \qquad j = j_0, \ldots, k\,, \\ \sum_{i=1}^{k} c_i f(\xi_{p_i}) &= \sum_{i=1}^{k} c_i p_i\,. \end{aligned}$$

Hence $c_{j_0} = \ldots = c_k = 0$ which leads to the situation analogical to the case $\xi_{p_k} < 0$. Thus we see that (2.16) holds for every non-zero vector $\mathbf{c} \in R^k$ and the matrix (2.52) is regular. By means of this regularity we obtain that

$$0 < |\boldsymbol{\Sigma}| = |\mathbf{A} - \mathbf{B}\,\mathbf{B}'| = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{I_2} \end{vmatrix} = |\mathbf{A}||\mathbf{I_2} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}|$$

and also the matrix $\mathbf{I_2} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}$ is regular. This together with the result of excercise 2.9 in the section 1.b.8 in [16] means that

$$\boldsymbol{\Sigma}^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{B}'\mathbf{A}^{-1}\mathbf{B} - \mathbf{I_2})^{-1}\mathbf{B}'\mathbf{A}^{-1}$$

and the rest of the proof follows from Lemma 3.1. □

Proof of Theorem 2. 5. (I) Validity of this assertion follows from Theorem 2.2(I).

(II) Let us assume that (3.12), (3.15) hold. Similarly as in the proof of the previous theorem one can show that the conditions *(C 1) - (C 4)* are for $\theta = \vartheta$ fulfilled with

$$\mathbf{l}(x, \vartheta) = \begin{pmatrix} x \\ |x| - 1 \end{pmatrix}.$$

This according to Theorem 2.2 means that the assumptions of Theorem 2.1(I) are fulfilled and (2.62) holds.

Making use of (2.47), (2.48) one gets after some computation that

$$cov(\chi_{(-\infty,\xi_p\rangle}(x), x) = -f(\xi_p)(|\xi_p| + 1)\,, \qquad cov(x, |x| - 1) = 0\,.$$

Hence taking into account (3.23) and the equality $Var(\,x\,|\,L(0,1)\,) = 2$ it is easy to see that for $\theta = \vartheta$ in (2.10)

$$\mathbf{L} = \begin{pmatrix} 2\ ,\ 0 \\ \\ 0\ ,\ 1 \end{pmatrix}, \qquad \mathbf{C} = -\mathbf{D}\mathbf{B}\,, \qquad \mathbf{B} = \begin{pmatrix} |\xi_{p_1}| + 1\ ,\ \xi_{p_1} \\ \vdots \\ |\xi_{p_k}| + 1\ ,\ \xi_{p_k} \end{pmatrix},$$

where $\mathbf{D}$ is diagonal $k \times k$ matrix with the diagonal $f(\xi_{p_1}), \ldots, f(\xi_{p_k})$ and $\mathbf{\Psi}$ is described in (3.24). Substituting into (2.10) we easily obtain that (2.63) holds.

Let us assume that for the random vectors $\mathbf{s}$, $\mathbf{t}$ described in Theorem 2.1(III) the equality (3.25) holds $L(0,1)$ a.e.. Then letting x tend to $+\infty$ and to $-\infty$ we get that

$$-\sum_{i=1}^{k} c_i f(\xi_{p_i}) - \sum_{i=1}^{k} c_i f(\xi_{p_i})\xi_{p_i} = 0\,,$$
$$-\sum_{i=1}^{k} c_i f(\xi_{p_i}) + \sum_{i=1}^{k} c_i f(\xi_{p_i})\xi_{p_i} = 0\,.$$

Thus these sums equal zero and (3.25) implies that

$$\sum_{i=1}^{k} c_i \chi_{(-\infty,\xi_{p_i}\rangle}(x) = \sum_{i=1}^{k} c_i p_i\,,$$

from which validity of

$$c_1 = \ldots = c_k = 0$$

can be easily proved and regularity of $\mathbf{\Sigma}$ follows from Theorem 2.1(III). The rest of the proof is obvious. □

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