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ON HODGES-LEHMANN OPTIMALITY OF LR TESTS[1]

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It is shown that the likelihood ratio test statistics are Hodges-Lehmann optimal for testing the null hypothesis against the whole parameter space, provided that certain regularity conditions are fulfilled. These conditions are verified for the non-singular normal, multinomial and Poisson distribution.

Let $\{\overline{P}_\gamma;\ \gamma \in \Xi\}$ be a family of probability measures, defined on $(X, \mathcal{F})$ by means of the densities $\{f(x,\gamma);\ \gamma \in \Xi\}$ with respect to a measure ν. If we denote the q-fold products

$$S = X^\infty \times \ldots \times X^\infty\,, \quad \mathcal{S} = \mathcal{F}^\infty \times \ldots \times \mathcal{F}^\infty\,, \quad \Theta = \Xi^q \tag{1}$$

then for $\theta = (\theta_1, \ldots, \theta_q) \in \Theta$ the corresponding product measure

$$P_\theta = \overline{P}_{\theta_1}^\infty \times \ldots \times \overline{P}_{\theta_q}^\infty\,, \tag{2}$$

defined on the σ-algebra $\mathcal{S}$, describes independent sampling from the q populations $(X, \mathcal{F}, \overline{P}_{\theta_j})$, $j = 1, \ldots, q$.

Throughout the paper we shall assume that

$$\emptyset \neq\ \Omega_0 \subset\ \Theta\,. \tag{3}$$

In describing asymptotic properties of tests of the null hypothesis we shall use the notation

$$\mathcal{P} = \left\{ p \in R^q;\ \sum_{j=1}^q p_j = 1\ \ and\ \ p_j > 0\ for\ all\ j \right\} \tag{4}$$

and for θ,$\theta^* \in \Theta$, $p \in \mathcal{P}$ we denote

$$K(\theta^*, \theta, p) = \sum_{j=1}^q p_j K(\theta_j^*, \theta_j)\,, \tag{5}$$

$$K(\Omega_0, \theta, p) = \inf\{\, K(\theta^*, \theta, p);\ \theta^* \in \Omega_0 \,\}\,, \tag{6}$$

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where $K(\theta_j^*, \theta_j) = K(\overline{P}_{\theta_j^*}, \overline{P}_{\theta_j})$ is the Kullback-Leibler information number.

We shall suppose that a test φ_u of Ω_0 against $\Theta - \Omega_0$ depends on

$$s = (\{x_j^{(1)}\}_{j=1}^{\infty}, \ldots, \{x_j^{(q)}\}_{j=1}^{\infty}) \in S$$

through

$$x^{(u)} = (y(1, n_u^{(1)}), \ldots, y(q, n_u^{(q)})) \tag{7}$$

only, where

$$y(j, n_u^{(j)}) = (x_1^{(j)}, \ldots, x_{n_u^{(j)}}^{(j)}) \tag{8}$$

is a sample from the j-th population.The sample sizes will be subjected to the following assumption, which in the one sample case $q = 1$ simply means that the sample size n tends to infinity.

(C 1) With the notation

$$n_u = \sum_{j=1}^{q} n_u^{(j)}, \qquad p_u^{(j)} = n_u^{(j)}/n_u \tag{9}$$

the relations

$$\lim_{u\to\infty} n_u = +\infty, \qquad \lim_{u\to\infty} p_u^{(j)} = p_j \in (0, 1\rangle, \quad j = 1, \ldots, q \tag{10}$$

hold.

If a test φ_u of Ω_0 against $\Theta - \Omega_0$ is based on (7) and for $\beta_u(\theta) = E_\theta(\varphi_u)$ the relation

$$\sup\{\lim_{u\to\infty} \beta_u(\theta^*);\ \theta^* \in \Omega_0\} = \alpha \in (0, 1) \tag{11}$$

holds, then according to Lemma 6.1 in [1] and Theorem 2.1 in [8] under validity of *(C 1)* for each parameter $\theta \in \Theta - \Omega_0$

$$\liminf_{u\to\infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] \geq -K(\Omega_0, \theta, p)\,. \tag{12}$$

We remark that an extension of this inequality to the case of stochastic processes and random fields can be found in [11].

In accordance with [4], [7] and [8] we shall say that the tests $\{\varphi_u\}$ are *Hodges-Lehmann optimal (H-L optimal) for testing* Ω_0 *against* $\Theta - \Omega_0$, if *(C 1)* and (11) imply that

$$\lim_{u\to\infty} \frac{1}{n_u} \log[1 - \beta_u(\theta)] = -K(\Omega_0, \theta, p) \tag{13}$$

for each $\theta \in \Theta - \Omega_0$.

The H-L optimality was investigated by Brown in [2], where testing Ω_0 against Ω_1 is replaced with testing Ω_0^* against Ω_1^*, and Ω_i^* is the closure of Ω_i in a set Θ^*, into which the original parameter set Θ is embedded. As pointed out in [2], p. 1208, the likelihood ratio statistics $T(x_1, \dots, x_n, \Omega_0, \Omega_1) = \log[L(x_1, \dots, x_n, \Omega_1)/L(x_1, \dots, x_n, \Omega_0)]$ can be essentially different from $T(x_1, \dots, x_n, \Omega_0^*, \Omega_1^*)$, which is in [2] proved to be optimal for the extended problem Ω_0^* against Ω_1^*.

In some particular cases was the H-L optimality of the LR tests proved in [8]. In the case of exponential families was this H-L optimality in the multisample case $q > 1$ proved in [7] under the conditions, including the following assumptions.

(D 1) The effective set

$$B = \{x; \sup_{\gamma} \log f(x, \gamma) < +\infty\} \tag{14}$$

is open.

(D 2) The arithmetic mean $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ *of the i.i.d. observations* $x_1, \dots, x_n$ *belongs to* B *with probability* 1 *for all* $n \geq N$ *and* $\gamma \in \Xi$.

These assumptions are in [7] imposed to assure existence of the MLE with probability 1 for all $n \geq N$. They are fulfilled by the family generated by the exponential reparametrization of the non-singular normal distributions, but for important exponential families - generated by the exponential reparametrization of the Poisson or the multinomial distributions - the set (14) is closed, and the MLE does not exist with positive probability for all n. Moreover, the null hypothesis Ω_0 is in [7] assumed to have the property that the function (25) of the parameter θ^* is continuous. The aim of this paper is to present regularity assumptions, fulfilled by the non-singular normal, the Poisson and the multinomial distributions. As it can be seen from Theorem 1, the presented assumptions in difference from [7] facilitate a unified approach to the H-L optimality of the LR tests for all integers $q \geq 1$, without imposing any restrictions on value of the quantity $K(\theta)$ (defined in [7] on p. 7), or on $K(., \Omega_0, p)$.

(A I) There exists a σ*-compact metric space* Ξ_1 *such that* Ξ *is dense in* Ξ_1 *(i.e.* $\overline{\Xi} = \Xi_1$*), and the original system* $\{f(x, \gamma); \gamma \in \Xi\}$ *can be extended to a system* $\{f(x, \gamma); \gamma \in \Xi_1\}$ *of densities with respect to the same measure* ν *(by the extension we understand that* $f(x, \gamma)$ *coincides with the original density if* $\gamma \in \Xi$*). Moreover, the extended system* $\{\overline{P}_\gamma; \gamma \in \Xi_1\}$ *consists of mutually different probabilities, the Kullback-Leibler information quantity* $K(\theta^*, .)$ *is continuous on* Ξ_1 *for each* $\theta^* \in \Xi_1$ *(i.e.* $\gamma_n^* \to \gamma^*$ *implies* $K(\theta^*, \gamma_n^*) \to K(\theta^*, \gamma^*)$*), and* $f(x, \gamma)$ *is continuous on* Ξ_1 *for each* $x \in X$.

(A II) The function $K(., \theta)$ *is finite and continuous on* Ξ_1 *for each* $\theta \in \Xi$.

(A III) Let $\{\theta_n^*\}_{n=1}^{\infty}$ *be parameters from* Ξ_1, $\gamma \in \Xi$ *and*

$$\limsup_{n \to \infty} K(\theta_n^*, \gamma) < +\infty \, .$$

(a) If $\theta \in \Xi$, *then* $\limsup_{n \to \infty} K(\theta_n^*, \theta) < +\infty$.

(b) The sequence $\{\theta_n^*\}_{n=1}^{\infty}$ *possesses a limit point in* Ξ_1 *and if the parameters* $\{\theta_n\}_{n=1}^{\infty}$ *belonging to* Ξ *are such that* $\limsup_{n\to\infty} K(\theta_n^*, \theta_n) < +\infty$ *, then also the sequence* $\{\theta_n\}_{n=1}^{\infty}$ *possesses a limit point in* Ξ_1 *.*

(c) Let $\lim_{n\to\infty} \theta_n^* = \theta^*$*. If* $\{\theta_n\}_{n=1}^{\infty}$ *is a sequence of parameters from* Ξ *and* $\lim_{n\to\infty} \theta_n = \theta$*, then* $\liminf_{n\to\infty} K(\theta_n^*, \theta_n) \geq K(\theta^*, \theta)$ *.*

(A IV) There exist measurable sets $A_n \subset X^n$ *, an integer* N *and measurable mappings* $\hat{\theta}_n : A_n \to \Xi_1$ *,* $g_n : A_n \to R$ *such that for each* $\gamma \in \Xi$ *and* $n \geq N$

$$\overline{P}_{\gamma}(A_n) = 1 \tag{15}$$

and on the set A_n *with the notation* $L(x_1, \ldots, x_n, \gamma) = \prod_{i=1}^{n} f(x_i, \gamma)$ *the equality*

$$\log L(x_1, \ldots, x_n, \gamma) = g_n(x_1, \ldots, x_n) - nK(\hat{\theta}_n, \gamma) \tag{16}$$

holds.

(A V) With the notation from (A IV) and

$$p_u = (p_u^{(1)}, \ldots, p_u^{(q)})\,, \tag{17}$$

for $\theta = (\theta_1, \ldots, \theta_q) \in \Xi^q$ *let (cf.* (8) *)*

$$K\Big(\hat{\theta}, \theta, p_u\Big) = \sum_{j=1}^{q} p_u^{(j)} K\Big(\,\hat{\theta}_{n_u^{(j)}}(y(j, n_u^{(j)})), \theta_j\,\Big)\,. \tag{18}$$

If (10) holds and $\theta \in \Xi^q$ *, then for every real* t

$$P_{\theta}[\, n_u K(\hat{\theta}, \theta, p_u) \geq t\,] \leq \exp[-t + h_u(t)] \tag{19}$$

where

$$\limsup_{u\to\infty} \frac{h_u(t_u)}{n_u} = 0 \tag{20}$$

whenever the inequality

$$\limsup_{u\to\infty} \frac{t_u}{n_u} < +\infty \tag{21}$$

holds.

We recall that a metric space is said to be σ-compact, if it can be expressed as a countable union of its compact subsets. The reader can easily verify that both this property of Ξ_1 and the continuity of $f(x, .)$, which are postulated in *(A I)*, imply measurability of (22) for any non-empty set $\Omega \subset \Theta = \Xi^q$.

The regularity assumptions imposed for the multisample case in [7] include only the exponential families, for which the MLE exists with probability 1 for all $n \geq N$. Instead of the exponential reparametrization we use an enlargement of the class of probabilities. Since the Kullback-Leibler information number is non-negative, the axiom *(A IV)* guarantees that for all $n \geq N$ there exists with probability 1 a MLE $\hat{\theta}_n$ of the unknown parameter $\gamma \in \Xi$, but taking values in the enlarged parameter set Ξ_1, and having the property that the likelihood functions computed for the original parameters $\gamma \in \Xi$ can be expressed in the specific postulated way. We remark that for the exponential families, satisfying the assumptions *(D 1)* and *(D 2)*, imposed in [7], the assumption *(A IV)* is fulfilled with $\Xi_1 = \Xi$. Validity of *(A IV)* makes possible to use the inequality (2.8) from [3] in the proof of H-L optimality of LR tests.

If we put for $\Omega \subset \Theta$

$$L(x^{(u)}, \Omega) = \sup\{ \prod_{j=1}^{q} \prod_{i=1}^{n_u^{(j)}} f(x_i^{(j)}, \theta_j);\ \theta = (\theta_1, \ldots, \theta_q) \in \Omega \}\ , \tag{22}$$

then by the likelihood ratio test statistics T_u for testing Ω_0 against $\Theta - \Omega_0$ we shall understand the statistics

$$T_u = 2\ \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)}\,. \tag{23}$$

In the proof of the theorem on H-L optimality of (23) the following lemma will be used.

Lemma 1. *Let the assumptions (A I)-(A III) be fulfillled, (3) hold and for $\delta \geq 0$ let*

$$D(\delta, p) = \{\, \theta^* \in \Xi_1^q; K(\theta^*, \Omega_0, p) \leq \delta \,\}\,, \tag{24}$$

where

$$K(\theta^*, \Omega_0, p) = \inf\{\, K(\theta^*, \tilde{\theta}, p);\ \tilde{\theta} \in \Omega_0 \,\}\,. \tag{25}$$

If $\lim_{u\to\infty} \delta_u = 0$, $\lim_{u\to\infty} p_u = p \in \mathcal{P}$, then for each $\theta \in \Theta$ with the notation (6)

$$d_u = K[D(\delta_u, p_u), \theta, p_u] \tag{26}$$

is a real number and

$$\liminf_{u\to\infty} d_u \geq K(\Omega_0, \theta, p)\,. \tag{27}$$

Proof. Since $d_u \leq K(\theta^*, \theta, p_u)$ whenever $\theta^* \in \Omega_0$, taking into account *(A II)* we see that

$$\limsup_{u\to\infty} d_u < +\infty\,. \tag{28}$$

Since the sets (24) are non-empty, there exist $\theta_u^* \in D(\delta_u, p_u)$ such that

$$d_u \leq K(\theta_u^*, \theta, p_u) \leq d_u + u^{-1}\,. \tag{29}$$

Let $\theta_u^{(0)} \in \Omega_0$ be such that

$$K(\theta_u^*, \Omega_0, p_u) \leq K(\theta_u^*, \theta_u^{(0)}, p_u) \leq K(\theta_u^*, \Omega_0, p_u) + u^{-1}\,. \tag{30}$$

Let us choose an increasing sequence $\{u_v\}_{v=1}^{\infty}$ of positive integers such that

$$\liminf_{u\to\infty} d_u = \lim_{v\to\infty} d_{u_v}\,. \tag{31}$$

Fixing $\gamma_0 \in \Omega_0$ and utilizing (28)-(30), $p_u \to p$ and *(A III)(a)* we get

$$\limsup_{u\to\infty} K(\theta_u^*, \theta_u^{(0)}, p_u) \le \limsup_{u\to\infty} K(\theta_u^*, \gamma_0, p_u) < +\infty$$

which together with *(A III)(b)* means that there exists a subsequence $\{u_{v_t}\}_{t=1}^{\infty}$ and points $\theta^* \in \Xi_1$, $\theta^{(0)} \in \overline{\Omega}_0$ such that with the notation

$$z_t = u_{v_t}\,, \qquad \gamma_t^* = \theta_{z_t}^*\,, \qquad \gamma_t^{(0)} = \theta_{z_t}^{(0)}$$

for $t \to \infty$

$$\gamma_t^* \to \theta^*\,, \quad \gamma_t^{(0)} \to \theta^{(0)}\,.$$

If $\tilde{\theta} \in \Omega_0$, then *(A II)*, (30) and *(A III)(c)* yield

$$K(\theta^*, \tilde{\theta}, p) = \lim_{t\to\infty} K(\gamma_t^*, \tilde{\theta}, p_{z_t}) \ge \limsup_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \ge$$

$$\ge \liminf_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t}) \ge K(\theta^*, \theta^{(0)}, p) \ge \; K(\theta^*, \overline{\Omega}_0, p)\,.$$

Since according to *(A I)*

$$K(\theta^*, \overline{\Omega}_0, p) = K(\theta^*, \Omega_0, p)\,, \tag{32}$$

obviously $K(\theta^*, \Omega_0, p) = \lim_{t\to\infty} K(\gamma_t^*, \gamma_t^{(0)}, p_{z_t})$. Combining this with (30) and $\theta_u^* \in D(\delta_u, p_u)$ we see that $K(\theta^*, \Omega_0, p) = 0$. Utilizing *(A III)(b), (A I)* we obtain that $\theta^* \in \overline{\Omega}_0$ and by virtue of (31), (29), *(A II)*

$$\liminf_{u\to\infty} d_u = \lim_{t\to\infty} K(\gamma_t^*, \theta, p_{z_t}) = K(\theta^*, \theta, p) \ge \; K(\overline{\Omega}_0, \theta, p)\,.$$

The inequality (27) now follows from *(A II)*. □

Theorem 1. *Let a family of probabilities* $\{\,\overline{P}_\gamma; \gamma \in \Xi\,\}$*, determined by densities* $\{f(x,\gamma);$ $\gamma \in \Xi\,\}$ *with respect to a measure* ν *be such that the assumptions (A I) - (A V) are fulfilled. Let*

$$\emptyset \ne \Omega_0 \subset \Theta = \Xi^q \tag{33}$$

and with the notation (23)

$$\varphi_u(x^{(u)}) = \begin{cases} 1 & T_u > t_u \\ \\ 0 & T_u \le t_u\,. \end{cases} \tag{34}$$

If validity of (C 1) and (11) implies that the critical constants t_u *in (34) can be chosen in* *such a way that*

$$\limsup_{u\to\infty} t_u < +\infty \tag{35}$$

then the tests (34) are H-L optimal.

Proof. We shall proceed analogously as in the proof of Theorem 4.5 in [7]. Let us assume that both *(C I)* and (11) hold. In accordance with *(A IV)* let

$$B_u = A_{n_u^{(1)}} \times \ldots \times A_{n_u^{(q)}}$$

and for $x^{(u)} \in B_u$ put

$$\hat{\theta}_{(u)}(x^{(u)}) = \left(\hat{\theta}_{n_u^{(1)}}(y(1, n_u^{(1)})), \ldots, \hat{\theta}_{n_u^{(q)}}(y(q, n_u^{(q)}))\right) . \tag{36}$$

Making use of *(A IV), (A II)* we get for $u \geq u_0$ with the notation (24) that

$$P_\theta[2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\Omega_0)} \leq t_u] = P_\theta[2n_u K(\hat{\theta}_{(u)}, \Omega_0, p_u) \leq t_u] =$$

$$= P_\theta[\hat{\theta}_{(u)} \in D(\delta_u, p_u)] \tag{37}$$

where $\delta_u = t_u/2n_u$. According to the assumptions $\lim_{u\to\infty} \delta_u = 0$. Utilizing the first inequality in (2.8) in [3] and applying the assumption *(A V)* we see that for $u \geq u_1$

$$P_\theta[\hat{\theta}_{(u)} \in D(\delta_u, p_u)] \leq P_\theta[K(\hat{\theta}_{(u)}, \theta, p_u) \geq d_u] \leq \exp[-n_u d_u + o_u] \tag{38}$$

where $\lim_{u\to\infty} o_u/n_u = 0$. Combining (37) and (38) with (27) we get

$$\limsup_{u\to\infty} \frac{1}{n_u} \log P_\theta[\varphi_u = 0] \leq -K(\Omega_0, \theta, p)$$

and (13) follows from (12). □

According to Theorem 1 the LR tests (34) based on (23) are H-L optimal for arbitrary non-empty set $\Omega_0 \subset \Xi^q$, having the property (35). However, if $\theta^* \in \Omega_0$, then

$$2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\Omega_0)} \leq \tilde{T}_u(x^{(u)},\theta^*)\,, \quad \tilde{T}_u(x^{(u)},\theta^*) = 2\log\frac{L(x^{(u)},\Theta)}{L(x^{(u)},\theta^*)}\,. \tag{39}$$

If Ξ is an open subset of R^m and the densities $\{f(x,\gamma);\, \gamma \in \Xi\}$ satisfy regularity conditions on partial derivatives, then the MLE $\hat{\theta}_n$ of the unknown parameter from Ξ has the property that $\mathcal{L}(\sqrt{n}(\hat{\theta}_n - \gamma)|P_\gamma) \to N(0, J^{-1}(\gamma))$, where $J(\gamma)$ is the Fisher information matrix, and according to the well-known classical results under validity of $(C1)$ for $u \to \infty$ the weak convergence of distributions

$$\mathcal{L}(\tilde{T}_u(x^{(u)},\theta^*)|P_{\theta^*}) \to \chi^2_{mq} \tag{40}$$

holds, where χ^2_{mq} denotes the chi-square distribution with mq degrees of freedom. From (39) and (40) one easily finds out, that validity of $(C1)$ and (11) implies (35). This fact will be used in the proofs of theorems in the further text.

Let $k > 1$ be an integer and $a = (k-1)k/2$. Let us put $m = 2k + a$ and denote

$$\Xi = \{\, \gamma = (\mu', \sigma_1, \ldots, \sigma_k, \rho')' \in R^m;\ \mu \in R^k, \min \sigma_i > 0, \rho \in R^a,\ R(\rho) \gg 0 \,\} \tag{41}$$

the set of parameters of the non-singular k-dimensional normal distributions, i.e. μ is the vector of means, σ_i^2 are the variances, $\rho = (\rho_{12}, \ldots, \rho_{k-1k})'$ are the correlation coefficients and $R(\rho)$ is the symmetric matrix with $R(\rho)_{ij} = \rho_{ij}$ if $i < j$, and $R(\rho)_{ii} = 1$, the notation $R(\rho) \gg 0$ means that the matrix $R(\rho)$ is positive definite. For $\gamma \in \Xi$ we shall denote by $V(\gamma)$ the covariance matrix of the corresponding normal distribution and $f(x, \gamma)$ its density. With this notation the following theorem holds.

Theorem 2. *If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H-L optimal for testing Ω_0 against $\Theta - \Omega_0$.*

Proof. Since (40) holds with $m = \frac{k(k+3)}{2}$, the tests (34) have the property (35). We shall prove that also the assumptions *(A I) - (A V)* are fulfilled.

Let us denote $\Xi_1 = \Xi$. For $\gamma, \gamma^* \in \Xi$

$$K(\gamma\,, \gamma^*) = \frac{1}{2}(\mu - \mu^*)' V(\gamma^*)^{-1}(\mu - \mu^*) + \frac{1}{2}\mathrm{tr}[V(\gamma)V(\gamma^*)^{-1}] + \frac{1}{2}\log\frac{|V(\gamma^*)|}{|V(\gamma)|} - \frac{k}{2}$$

and *(A I), (A II)* obviously hold.

According to the inequality (3.31) in [6]

$$K(\gamma^*, \gamma) \geq \frac{1}{2}(\mu^* - \mu)' V(\gamma)^{-1}(\mu^* - \mu) + \frac{1}{2}\sum_{j=1}^{k} g\Big(\frac{\lambda_j(\gamma)}{\lambda_j(\gamma^*)}\Big) - \frac{k}{2}$$

where $\lambda_1(\gamma) \geq \ldots \geq \lambda_k(\gamma)$ are characteristic roots of $V(\gamma)$ and $g(z) = z^{-1} + \log z$. Since $g(z)$ attains its minimum in $z = 1$ and $g(z) \longrightarrow +\infty$ if either $z \to +\infty$ or $z \to 0^+$, the assumption *(A III)* holds.

The assumption *(A V)* can be proved either similarly as Theorem 2.1 in [3], or after the exponential reparametrization by means of the Lemma 4.4 in [7]. Since *(A IV)* holds with the usual MLE $\hat{\theta}_n$ and with the set A_n of the n-tuples $(x_1, \ldots, x_n) \in (R^k)^n$ for which the matrix $\sum_{j=1}^n (x_j - \overline{x})(x_j - \overline{x})'$ is positive definite, the assumptions of Theorem 1 are fulfilled. □

As we have already noted, this Theorem 2 can be applied to testing the null hypothesis (3) on parameters of the non-singular normal distribution by means of the test statistics (23). Since in some situations the likelihood ratio test statistics (23) are expressible as monotone transformations $Z_u(T_u^*)$ of some usually used test statistics T_u^*, in such a case also H-L optimality of T_u^* is established. In the following example the statistics T_u^* have not this property.

Example. Testing equality of covariances. Let us denote in accordance with (1) and (41)

$$\Omega_0 = \{\theta = (\theta_1, \ldots, \theta_q) \in \Theta;\ V(\theta_1) = \ldots = V(\theta_q)\} \tag{42}$$

the hypothesis that the covariance matrices of the q normal populations are equal. Let

$$\bar{x}_j,\ \hat{\Sigma}_j$$

denote the sample mean and the sample covariance matrix constructed from the sample drawn from the j-th population. If we put $S_j = n_u^{(j)} \hat{\Sigma}_j$, $S = \sum_{j=1}^{q} S_j$, then

$$T_u = 2 \log \frac{L(x^{(u)}, \Theta)}{L(x^{(u)}, \Omega_0)} = \log \tilde{T}_u\ , \qquad \tilde{T}_u = \left| \frac{1}{n_u} S \right|^{n_u} \Big/ \prod_{j=1}^{q} \left| \frac{1}{n_u^{(j)}} S_j \right|^{n_u^{(j)}} .$$

As pointed out in [10], p.225, to obtain an unbiased test, instead of $\tilde{T}_u$ the modified test statistic

$$T_u^* = \left| \frac{1}{n_u - q} S \right|^{n_u - q} \Big/ \prod_{j=1}^{q} \left| \frac{1}{n_u^{(j)} - 1} S_j \right|^{n_u^{(j)} - 1} \tag{43}$$

is used. We shall prove the H-L optimality of the statistic T_u^*.

As it is shown in the proof of Theorem 3.1 in [6], if $\gamma \in \Xi$ and a is a real number, then there exist real numbers $0 < \varepsilon < \beta$ such that with the notation

$$B_n = \{\, (x_1, \ldots, x_n);\ \lambda_k(\hat{\Sigma}) \geq \varepsilon\,, \quad \lambda_1(\hat{\Sigma}) \leq \beta \,\}\,, \tag{44}$$

where $\lambda_k(\hat{\Sigma})$ is the smallest and $\lambda_1(\hat{\Sigma})$ is the largest characteristic root of $\hat{\Sigma}$, the inequality

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[1 - \overline{P}_\gamma(B_n)\right] \leq -a \tag{45}$$

holds. Let (10) hold. If $a = K(\Omega_0, \theta, p)$ and $B_n^{(j)}$ are the sets (44) satisfying (45) with $\gamma_j = \theta_j$ and $a_j^* = a/p_j$, then with the notation $X = R^k$, $C_u = B^{(1)}_{n_u^{(1)}} \times \ldots \times B^{(q)}_{n_u^{(q)}}$

$$\limsup_{u \to \infty} \frac{1}{n_u} \log P_\theta \Big[X^{n_u} - C_u\Big] \leq -a\,. \tag{46}$$

Under validity of the null hypothesis $\tilde{T}_u / T_u^* \to 1$ a. e., and according to [9], p. 404 the distributions $\mathcal{L}(\log \tilde{T}_u) \to \chi_v^2$. Thus if costants $\{t_u\}$ are such that the tests (34) based on $T_u = \log T_u^*$ satisfy (11), then (35) is fulfilled. Further, it is obvious that there exist an index u_1 and a positive real number M such that for all $u \geq u_1$ on the set C_u the inequality $\tilde{T}_u / T_u^* \leq M$ holds. Hence according to Theorem 2

$$\limsup_{u \to \infty} \frac{1}{n_u} \log P_\theta [C_u \cap \{\log T_u^* \leq t_u\}] \leq$$

$$\leq \limsup_{u \to \infty} \frac{1}{n_u} \log P_\theta [\, \log \tilde{T}_u \leq t_u + \log M] = -K(\Omega_0, \theta, p)\,.$$

Combining this with (46) we get

$$\limsup_{u\to\infty} \frac{1}{n_u} \log P_\theta[\log T_u^* \le t_u] \;\le\; -K(\Omega_0, \theta, p)\,,$$

which together with (12) means that the tests based on T_u^* are H-L optimal for testing the hypothesis (42) under the normality assumptions.

Let $X = \{1, ..., k\}$ be a finite set,

$$\Xi = \{\, (p_1, ..., p_{k-1})' \in R^{k-1};\ \min_i p_i > 0,\ \sum_{i=1}^{k-1} p_i < 1\} \tag{47}$$

and

$$f(x,p) = p_x\,, \qquad p_k = 1 - \sum_{j=1}^{k-1} p_j \tag{48}$$

denotes a density with respect to the counting measure μ on $(X, 2^X)$. With this notation the following theorem holds.

Theorem 3. *If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H-L optimal for testing Ω_0 against $\Theta - \Omega_0$.*

Proof. Since (40) holds with $m = k-1$, the tests (34) have the property (35). To prove *(A I) - (A V)*, let us denote

$$\Xi_1 = \overline{\Xi} \tag{49}$$

the closure of (47) and let $f(x,p)$ be the densities (48). Since

$$K(p, p^*) = \sum_{j=1}^{k} p_j \log \frac{p_j}{p_j^*}$$

where $0 \log \frac{0}{x} = 0$, the assumption *(A III)(c)* can be proved similarly as Lemma 4.4(a) in [5], and all the other statements in *(A I) - (A III)* can be easily proved by means of the compactness of Ξ_1.

The assumption *(A IV)* is fulfilled with $A_n = X^n$ and

$$\hat\theta_n = (\hat p_1, \ldots, \hat p_{k-1})'\,, \qquad \hat p_x = \frac{n_x}{n}$$

where n_x denotes the number of occurrences of x in $x_1, \ldots, x_n$.

Making use of the first equality in (48), the relation (2.4) in [5] and proceeding as in the proof of the inequality (2.10) in [5], we obtain, that *(A V)* holds.

Thus the assumptions of Theorem 1 are fulfilled and the proof is completed. □

Let $X = \{0, 1, 2, ...\}$,

$$\Xi = (0, +\infty) \tag{50}$$

and

$$f(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{51}$$

be density of the Poisson distribution $\overline{P}_\lambda$ with respect to the counting measure μ on $(X, 2^X)$. With this notation the following theorem holds.

Theorem 4. *If Ω_0 is a non-empty subset of $\Theta = \Xi^q$ and T_u are the statistics (23), then the tests (34) are H-L optimal for testing Ω_0 against $\Theta - \Omega_0$.*

Proof. Since (40) holds with $m = 1$, the tests (34) have the property (35). To prove *(A I) - (A V)*, we denote

$$\Xi_1 = \langle 0\,, +\infty)\,.$$

Since for $\lambda,\ \lambda^* \in \Xi_1$

$$K(\lambda, \lambda^*) = \begin{cases} \lambda^* - \lambda + \lambda \log \dfrac{\lambda}{\lambda^*} & \lambda^* > 0 \\ +\infty & \lambda^* = 0,\ \lambda > 0 \end{cases}$$

where $0 \log \frac{0}{x} = 0$, the assumptions *(A I)-(A III)* hold. The assumption *(A IV)* holds with $A_n = X^n$ and $\hat{\theta}_n = n^{-1}\sum_{j=1}^n x_j$. Since the axiom *(A V)* can be proved either by means of the relation (6.22) in [6] or by means of the Lemma 4.3 in [7], the assumptions of Theorem 1 are fulfilled. □

REFERENCES

[1] R. R. Bahadur: Some Limit Theorems in Statistics. SIAM, Philadelphia, 1971.

[2] L. D. Brown: Non-local asymptotic optimality of appropriate likelihood ratio tests. Ann. Math. Statist. 42(1971), 1206-1240.

[3] D. G. Herr: Asymptotically optimal tests for multivariate normal distributions. Ann. Math. Statist. 38(1967), 1829–1844.

[4] J. L. Hodges and E. L. Lehmann: The efficiency of some nonparametric competitors of the t-test. Ann. of Math. Statist. 27(1956), 324–335.

[5] W. Hoeffding: Asymptotically optimal tests for multinomial distributions. Ann. Math. Statist. 36(1965), 369–408.

[6] F. Rublík: On optimality of the LR tests in the sense of exact slopes. II. Application to individual distributions. Kybernetika 25(1989), 117–135.

[7] S. Kourouklis: Hodges-Lehmann optimality of the likelihood ratio test in regular exponential families of distributions. TR. No 73, Dept. of Statistics, The Pennsylvania State University, 1987.

[8] S. Kourouklis: Hodges-Lehmann efficacies of certain tests in multivariate analysis and

regression analysis. Canadian J. Statist. 16(1988), 87-95.
[9] A. M. Kshirsagar: Multivariate Analysis. Marcel Dekker, New York, 1972.
[10] M.S. Srivastava and C. G. Khatri: An Introduction to Multivariate Statistics. North-Holland, New York, 1979.
[11] I. Vajda: Generalization of discrimination-rate theorems of Chernoff and Stein. Kybernetika 26(1990), 273–288.

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