A QUANTILE GOODNESS-OF-FIT TEST FOR CAUCHY DISTRIBUTION, BASED ON EXTREME ORDER STATISTICS*

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Abstract. A test statistic for testing goodness-of-fit of the Cauchy distribution is presented. It is a quadratic form of the first and of the last order statistic and its matrix is the inverse of the asymptotic covariance matrix of the quantile difference statistic. The distribution of the presented test statistic does not depend on the parameter of the sampled Cauchy distribution. The paper contains critical constants for this test statistic, obtained from 50 000 simulations for each sample size considered. Simulations show that the presented test statistic is for testing goodness-of-fit of the Cauchy distributions more powerful than the Anderson-Darling, Kolmogorov-Smirnov or the von Mises test statistic.

Keywords: sample quantiles; chi-squared statistics; goodness-of-fit; Cauchy distribution

MSC 1991: 62G10, 62G30

1. INTRODUCTION

Suppose that $x_1, \ldots, x_n$ is a random sample. The null hypothesis, that it is drawn from a Cauchy distribution, is recommended in the Section 4.14 of [1] to be tested by means of the Anderson-Darling test statistic

$$A^2 = n \int_{-\infty}^{+\infty} \frac{\left(F_n(x) - \hat{F}(x)\right)^2}{\hat{F}(x)(1-\hat{F}(x))} \, \mathrm{d}\hat{F}(x) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log\left(Z^{(i)}(1 - Z^{(n+1-i)})\right), \tag{1.1}$$

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or by means of the von Mises test statistic

$$W^2 = n\int_{-\infty}^{+\infty} \left(F_n(x) - \hat{F}(x)\right)^2 \mathrm{d}\hat{F}(x) = \sum_{i=1}^{n} \left(Z^{(i)} - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}\,. \tag{1.2}$$

Here

$$F_n(x) = \frac{1}{n}\mathrm{card}\,\{j;\ j \le n,\ x_j \le x\,\} \tag{1.3}$$

is the empirical distribution function, $Z^{(i)}$ are the order statistics computed from $Z_i = F(x_i, \hat{\theta})$,

$$F(x,\theta) = F(\frac{x-\mu}{\sigma})\,, \qquad F(t) = \frac{1}{2} + \frac{1}{\pi}\arctan(t) \tag{1.4}$$

is the distribution function of the Cauchy distribution with the parameter $\theta = (\mu, \sigma)$, the estimator of the distribution function $\hat{F} = F(x, \hat{\theta})$, and $\hat{\theta} = (\hat{\mu}, \hat{\sigma})\,'$ is the estimator of the unknown parameter, computed by means of the order statistics $x_n^{(i)}$ from the formulas

$$\hat{\mu} = \sum_{i=1}^{n} g_{ni} x_n^{(i)}\,, \qquad g_{ni} = \frac{1}{n} G\Big(\frac{i}{n+1}\Big)\,, \tag{1.5}$$

$$G(u) = \frac{\sin(4\pi(u-0.5))}{\tan(\pi(u-0.5))} = -4\sin^2(\pi u)\cos(2\pi u)\,,$$

$$\hat{\sigma} = \sum_{i=1}^{n} c_{ni} x_n^{(i)}\,, \qquad c_{ni} = \frac{1}{n} J\Big(\frac{i}{n+1}\Big)\,, \tag{1.6}$$

$$J(u) = \frac{8\tan(\pi(u-0.5))}{\sec^4(\pi(u-0.5))} = -8\cos(\pi u)\sin^3(\pi u)\,.$$

This null hypothesis can be tested also by means of the Kolmogorov-Smirnov test statistic

$$KS = \sqrt{n}\sup_x |F_n(x) - \hat{F}(x)| = \sqrt{n}\max\{D_n^+\,,\ D_n^-\}\,, \tag{1.7}$$

where

$$D_n^+ = \max\{\,\frac{i}{n} - Z^{(i)};\ i = 1, \ldots, n\,\}\,, \qquad D_n^- = \max\{\,Z^{(i)} - \frac{i-1}{n};\ i = 1, \ldots, n\,\}\,.$$

As has been observed in [10], for the constants in (1.5) the equality $\sum_i g_{ni} = 1$ does not hold and therefore the estimate $\hat{\mu}$ is not equivariant. Consequently, the distribution of A^2 based on the trigonometric estimates (1.5), (1.6), depends on the parameter of the sampled Cauchy distribution and the upper tail percentage points for A^2 given in Table 4.26 on page 163 of [1] are not suitable for testing goodness-of-fit for the Cauchy distribution when the sample size is moderate and the parameter θ in (1.4) is unknown. This follows from the results of simulation in [10], which shows that for A^2 and values of the location parameter considered ibidem the nominal significance level is far below the actual size of the test. As can be seen from Table 1, in the case of the von Mises test statistic the situation is the same and when the scale parameter is fixed, the increase of the location parameter μ shifts the actual size of the test towards 1. We remark that the following table is computed from $N = 2000$ trials for each sample size and the upper tail percentage points $v(\alpha, n)$ are taken from Table 4.26 on the page 163 in [1]; all simulations considered in this paper were carried out by means of MATLAB, version 4.2c.1.

	$n = 10$				
μ	0	10	20	30	60
$P(W^2 > v(0.1, n))$	0.100	0.36	0.71	0.86	0.96
$P(W^2 > v(0.05, n))$	0.050	0.14	0.49	0.71	0.91
$P(W^2 > v(0.025, n))$	0.025	0.06	0.28	0.54	0.84

	$n = 30$				
μ	0	10	20	30	60
$P(W^2 > v(0.1, n))$	0.100	0.25	0.71	0.92	0.99
$P(W^2 > v(0.05, n))$	0.050	0.08	0.38	0.75	0.98
$P(W^2 > v(0.025, n))$	0.025	0.03	0.08	0.33	0.91

	$n = 50$				
μ	0	10	20	30	60
$P(W^2 > v(0.1, n))$	0.100	0.24	0.68	0.92	1
$P(W^2 > v(0.05, n))$	0.050	0.11	0.44	0.80	0.99
$P(W^2 > v(0.025, n))$	0.025	0.04	0.17	0.53	0.98

Table 1. Simulation of the probability of the type I error for the von Mises statistic W^2 based on the trigonometric location and scale estimates (1.5), (1.6).

Since for the Kolmogorov-Smirnov test statistic based on the estimates (1.5), (1.6) the situation is similar, the results of simulations are not included into the text in this case.

One apparent remedy for this situation is the use of some modified scores g_{ni}^* in the estimator (1.5) instead of the original ones, e.g., $g_{ni}^* = g_{ni}/\sum_j g_{nj}$ or $g_{ni}^* = g_{ni} + (1 - \sum_j g_{nj})/n$, but simulations show that this choice of the estimators yields test statistics which have not good performance under the alternatives. As it is well known, the Cauchy distributions are heavy-tailed, the sample median $\hat{\mu}$ is insensitive to the extreme values and since for (1.6) the equality $\hat{\sigma} = \sum_i |c_{ni}(x_n^{(i)} - \hat{\mu})|$ holds, from now on we consider only the case, when the test statistics (1.1), (1.2), (1.7) are based on the sample median

$$\hat{\mu} = \begin{cases} x_n^{(k+1)} & n = 2k+1\,, \\ \dfrac{x_n^{(k)} + x_n^{(k+1)}}{2} & n = 2k\,, \end{cases} \tag{1.8}$$

and on the trigonometric scale estimate (1.6). Percentage points of the statistics A^2, W^2, $T = Q(n, 0.1, 0.9)$ and KS, defined by the equalities $P(A^2 > a(0.05, n)) = 0.05$, $P(W^2 > w(0.05, n)) = 0.05$, $P(T > t(0.05, n)) = 0.05$, $P(KS > ks(0.05, n)) = 0.05$ and obtained from N=50 000 simulations for each considered sample size n, are given in the following table; the statistic $T = Q(n, 0.1, 0.9)$ is defined by means of the formula (2.5) of the next section.

n	14	16	18	20	25	35
$a(0.05, n)$	1.3361	1.3079	1.2890	1.2528	1.2276	1.1353
$w(0.05, n)$	0.2336	0.2238	0.2168	0.2058	0.1971	0.1743
$t(0.05, n)$	16.0122	13.2702	10.8749	13.5259	11.6821	9.8522
$ks(0.05, n)$	1.1091	1.1022	1.0881	1.0736	1.0554	1.0043

Table 2. Five percent upper tail percentage points of the statistics A^2, W^2, T and KS based on the median and on the trigonometric scale estimate.

These constants are used as a tool in the simulation study, presented in Table 4 of this paper.

Since the construction of the statistics (1.1), (1.2) and (1.7) is based on the empirical distribution function, they are often called the EDF statistics. The construction of the goodness-of-fit tests can be also carried out by applying disparities (divergence measures) of discrete probability vectors to appropriate partitions of the sample space. This approach was investigated in various publications; results, covering a general class of disparity measures and concerning goodness-of-fit tests based on a parameter estimator possessing a special type of the asymptotical representation, are derived in [8]. In the next section of this paper a new test statistic is constructed by means of the results related to the quantile test statistic.

2. TEST STATISTIC AND POWER COMPARISONS

The performance of the test statistics mentioned in the previous part of the paper will be in this section compared with the power of a test, which will be derived from the quantile test statistic. Here by the quantile test statistic one understands a chi-squared type statistic based on the difference between fixed chosen value of the probability and its estimate, obtained by plugging an estimator into the parameter value of the underlying theoretical probability. This approach has been a subject of various publications; a list of classical papers on this topic can be found on p. 129 of [3]. These papers assumed validity of the Rao-Cramér type regularity conditions on densities and utilized parameter estimators asymptotically equivalent to the maximum likelihood estimator. In a general setting, under mild regularity conditions on densities and with the underlying estimators assumed only to be asymptotically linearizable, was this idea studied in [9].

The quantile goodness-of-fit test statistic and its asymptotic behaviour can be in the case of the Cauchy distribution, when the parameter is estimated with (1.8), (1.6), described briefly as follows. Suppose that $k \geq 1$ is a fixed integer, $0 < p_1 < \ldots < p_k < 1$ are fixed real numbers and (cf. (1.3))

$$\hat{\xi}_{p_i,n} = \inf \{t;\, F_n(t) \geq p_i \}$$

denotes the p_ith sample quantile. Further, put (cf. (1.8), (1.6), (1.4))

$$\hat{\theta} = (\hat{\mu}, \hat{\sigma})' \,, \qquad \boldsymbol{\Delta}_n(p_1, \ldots, p_k) = \Big(F(\hat{\xi}_{p_1,n}, \hat{\theta}) - p_1, \ldots, F(\hat{\xi}_{p_k,n}, \hat{\theta}) - p_k\Big)' . \tag{2.1}$$

It is proved in [10] that

$$\hat{\sigma} - 1 = \frac{1}{n} \sum_{j=1}^{n} \psi(x_j) + \mathcal{O}_P(n^{-1}) \,, \qquad \psi(x) = -2\cos(2\arctan(x)) \,,$$

provided that the sample is drawn from the Cauchy $C(0,1)$ distribution. This asymptotic linearity together with the known asymptotic representation of the sample median imply that for such a sampling

$$\hat{\theta} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{l}(x_j) + o_P(n^{-1/2}) \,, \qquad \boldsymbol{l}(x) = \begin{pmatrix} \pi \mathrm{sign}(x)/2 \\ \\ -2\cos(2\arctan(x)) \end{pmatrix} ,$$

$$\sqrt{n} \left[\hat{\theta} - (0,1)' \right] \xrightarrow{\mathcal{L}} N(0, \boldsymbol{L}) \,, \qquad \boldsymbol{L} = \begin{pmatrix} \pi^2/4 \,, & 0 \\ \\ 0 \,, & 2 \end{pmatrix} .$$

These facts together with the results from [9] are in [10] employed in proving that the random vector $\sqrt{n}\mathbf{\Delta}_n(p_1,\ldots,p_k)$ is asymptotically $N_k(\mathbf{0},\mathbf{\Sigma})$ normal with the asymptotic covariance $k\times k$ matrix

$$\mathbf{\Sigma} = \mathbf{\Sigma}[k,p_1,\ldots,p_k] = \mathbf{A}+\mathbf{G}\,, \tag{2.2}$$

where

$$\mathbf{A} = (a_{ij})_{i,j=1}^{k}\,, \qquad a_{ij} = \min\{p_i,p_j\}\,(1-\max\{p_i,p_j\})\,, \tag{2.3}$$

$$\begin{aligned} \mathbf{G} &= (g_{ij})_{i,j=1}^{k}\,, \\ g_{ij} &= \frac{\sin^2(p_i\pi)\sin^2(p_j\pi)}{4} - \frac{sin^2(p_i\pi)}{2}\min\{p_j,1-p_j\} \\ &- \frac{sin^2(p_j\pi)}{2}\min\{p_i,1-p_i\} - \frac{\sin(2p_i\pi)\sin(2p_j\pi)}{2\pi^2}\,, \end{aligned} \tag{2.4}$$

provided that the random samples $x_1,\ldots,x_n$ are drawn from a Cauchy distribution. Further, if $p_i\neq 0.5$ for all $i=1,\ldots,k$, then according to Theorem 1.2 of [10] the matrix $\mathbf{\Sigma}[k,p_1,\ldots,p_k]$ is regular and the test statistic

$$Q(n,p_1,\ldots,p_k) = n\,\mathbf{\Delta}_n(p_1,\ldots,p_k)'\,\mathbf{\Sigma}[k,p_1,\ldots,p_k]^{-1}\,\mathbf{\Delta}_n(p_1,\ldots,p_k)\,, \tag{2.5}$$

where n denotes the sample size, has for $n>1$ an exact null distribution (i.e., the distribution of (2.5) does not depend on the parameter of the sampled Cauchy distribution). Thus under these assumptions the statistic (2.5) is asymptotically χ^2_k distributed and the null hypothesis H_0 that the random sample $x_1,\ldots,x_n$ comes from a Cauchy distribution, can be tested by means of the test rejecting H_0 for large values of (2.5). This rule is a chi-square goodness-of-fit test, and, as it is observed on. pp. 91 and 92 of [1], this class of tests usually does not posses large power and is less efficient than EDF tests. As the simulations in [10] for $Q(n,0.1,0.9)$ show, such a lack of power turns out to occur also for testing the goodness-of-fit with the Cauchy distribution, because an acceptable power is recorded only for sample sizes about 30 – 40 (the choice of larger number k of the partitioning points would only cause "an increase of the noise" and the quantile test would exert its power only for even larger values of the sample size). Probably one of the causes of this worse performance is not only a slower effect of the chi-square asymptotic, but also the approach consisting in keeping p_i's fixed, and for this reason we will choose the values of p_i's in dependence on the sample size.

As is well known (e.g., cf. Theorem d on p. 39 of [5]), the mean of $F(x_n^{(i)},\theta)$ equals $i/(n+1)$ provided that θ is the true value of the parameter of the sampled Cauchy distribution, and since heavy tails are typical for the Cauchy distribution, we propose to test the hypothesis H_0 by means of the test statistic $Q_n = Q(n, 1/(n+1), n/(n+1))$, which uses the first and the last order statistic in the role of the estimator of the $1/(n+1)$th and $n/(n+1)$th quantile, respectively. The limiting distribution of Q_n under validity of H_0 is unknown. Even though heureistic construction of the statistic Q_n uses asymptotically precise results from [10], in this case the quantiles involved do not obey the rule of convergence of their arguments to limits from the open interval $(0, 1)$. Hence the asymptotic distribution of $\mathbf{\Delta}_n(1/(n+1), n/(n+1))$ cannot be derived by means of the general assertions from [9] or their implementation in [10], and the application of the formulas from [10] on asymptotic covariances is not substantiated by the theory developed in the papers mentioned. Still, as will be seen from the results of a simulation study, the resulting statistic turns out to yield a powerful goodness-of-fit test for the Cauchy distribution.

In accordance with (2.5) the proposed test statistic

$$Q(n, 1/(n+1), n/(n+1)) = n\mathbf{\Delta}_n{}' \mathbf{\Sigma}[2, 1/(n+1), n/(n+1)]^{-1}\mathbf{\Delta}_n\,, \tag{2.6}$$

where

$$\mathbf{\Delta}_n = \left(F(x_n^{(1)}, \hat{\theta}) - \frac{1}{n+1}\,,\, F(x_n^{(n)}, \hat{\theta}) - \frac{n}{n+1}\right)'\,, \tag{2.7}$$

F is the distribution function from (1.4), $x_n^{(i)}$ denotes the ith order statistic, the parameter is estimated by the estimators (1.8), (1.6) and the matrix

$$\mathbf{\Sigma}[2, 1/(n+1), n/(n+1)]$$

is defined by (2.2) – (2.4). As has already been mentioned in the case of the statistic (2.5), the statistic (2.6) possesses for every sample size $n > 1$ a null distribution. Hence if the constants $q(\alpha, n)$ are chosen in an appropriate way then the rule

reject the null hypothesis that the sample is drawn from a Cauchy distribution, whenever $Q(n, 1/(n+1), n/(n+1)) > q(\alpha, n)$ (2.8)

is a test of this null hypothesis at the significance level α. Some values of these percentage points, obtained from a simulation study based on N=50 000 trials for each considered sample size n, are qiven in the following Table 3.

The proposed test is illustrated by the following example. It should be noted that the data used in this example are usually used in the literature on detecting outliers (e.g., in [2], p. 38 or in [4], pp. 54 – 55 and in the papers quoted ibidem).

n	5	6	7	8	9	10	11	12
$q(0.25, n)$	3.7450	3.5890	3.8511	3.6778	3.6308	3.4376	3.3747	3.2647
$q(0.1, n)$	5.8599	6.0991	7.2295	7.2331	7.7291	7.5023	7.7171	7.3992
$q(0.05, n)$	6.8867	7.7621	9.5586	9.9707	11.0639	11.0354	11.5535	11.4097
$q(0.025, n)$	7.4767	8.9927	11.2955	12.4928	14.2798	14.9293	15.9187	16.0462

n	13	14	15	16	17	18	19	20
$q(0.25, n)$	3.1164	3.0457	2.9397	2.8247	2.7674	2.6590	2.5917	2.5084
$q(0.1, n)$	7.3468	7.1040	7.1511	6.8661	6.8262	6.6339	6.3644	6.3644
$q(0.05, n)$	11.5410	11.0797	11.3721	10.8227	10.8357	10.6602	10.2758	10.1910
$q(0.025, n)$	16.3741	15.9191	16.4377	15.7432	15.9121	15.4159	15.2422	15.2261

n	21	22	23	24	25	26	27	28
$q(0.25, n)$	2.4880	2.3905	2.3488	2.3566	2.3049	2.2432	2.2513	2.1900
$q(0.1, n)$	6.3243	6.1329	6.1209	6.1256	6.0069	5.9015	5.8235	5.7266
$q(0.05, n)$	10.2032	9.8285	10.0805	9.9099	9.8597	9.6343	9.4169	9.2680
$q(0.025, n)$	15.0427	14.2978	14.7135	14.5984	14.4329	14.1077	13.7516	13.5003

n	29	30	31	32	33	34	35	36
$q(0.25, n)$	2.1469	2.1182	2.1002	2.1001	2.0714	2.0451	2.0284	2.0165
$q(0.1, n)$	5.6535	5.6805	5.6353	5.6498	5.5591	5.5330	5.4946	5.4367
$q(0.05, n)$	9.3652	9.3099	9.1658	9.1831	9.2205	9.0557	9.0115	8.9639
$q(0.025, n)$	13.9056	13.5609	13.5959	13.5158	13.7156	13.3241	13.2268	13.1754

n	37	38	39	40	41	42	43	44
$q(0.25, n)$	2.0171	1.9900	1.9793	1.9470	1.9509	1.9374	1.9493	1.9264
$q(0.1, n)$	5.3216	5.3213	5.3354	5.1714	5.2555	5.2086	5.2502	5.2453
$q(0.05, n)$	8.9527	8.7588	8.9510	8.7395	8.8636	8.8209	8.8489	8.6110
$q(0.025, n)$	13.3174	12.7726	13.1408	12.8612	13.0227	13.0458	13.1573	12.8705

n	45	46	47	48	49	50	55	60
$q(0.25, n)$	1.9199	1.9136	1.8945	1.9083	1.8767	1.8931	1.8269	1.8181
$q(0.1, n)$	5.1821	5.1932	5.0325	5.1197	5.0648	5.0861	4.9083	4.8850
$q(0.05, n)$	8.6398	8.6345	8.6431	8.5830	8.5850	8.5822	8.4739	8.3429
$q(0.025, n)$	12.8991	12.9040	12.7874	12.8345	13.0603	12.8995	12.7494	12.5112

n	65	70	75	80	90	100	125	150
$q(0.25, n)$	1.8072	1.7886	1.7716	1.7706	1.7583	1.7182	1.6916	1.6890
$q(0.1, n)$	4.8864	4.8174	4.8144	4.7495	4.8250	4.6629	4.6660	4.5717
$q(0.05, n)$	8.3571	8.2896	8.1596	8.1670	8.2777	8.1184	8.1222	7.9964
$q(0.025, n)$	12.6725	12.5569	12.4711	12.2007	12.4042	12.5744	12.5826	12.2524

Table 3. Upper tail percentage points of the quantile statistics $Q = Q(n, 1/(n+1), n/(n+1))$ based on the median (1.8) and on the trigonometric scale estimate (1.6).

Example. The following data consist of the individual deviations from the mean of 15 observations of the vertical semi-diameter of Venus.

-0.30	+0.48	+0.63	-0.22	+0.18
-0.44	-0.24	-0.13	-0.05	+0.39
+1.01	+0.06	-1.40	+0.20	+0.10

We shall employ the quantile test for testing the null hypothesis that this sample of 15 observations is generated by a Cauchy distribution (the fact that these values are deviations from the sample mean does affect the value and the distribution of the location and scale invariant statistic (2.6)). Since $n = 15$, for these data one obtains the involved matrices

$$\mathbf{A} = \begin{pmatrix} 0.0586 & 0.0039 \\ 0.0039 & 0.0586 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} -0.0094 & 0.0054 \\ 0.0054 & -0.0094 \end{pmatrix},$$

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} 21.0991 & -3.9954 \\ -3.9954 & 21.0991 \end{pmatrix},$$

the parameter estimates and the vector of differences

$$\hat{\mu} = 0.0600\,, \qquad \hat{\sigma} = 0.3184\,, \qquad \mathbf{\Delta}_n = (0.0058\,,\ -0.0404)'\,.$$

Thus the test statistic (2.6) attains in this case the value

$$Q(n, 1/(n+1), n/(n+1)) = 0.5565 < q(0.05, n) = 11.3721$$

and the null hypothesis is not rejected at the 5 per cent level. Simulations based on N=50 000 trials show that in this case the level attained (the P-value)

$$\hat{\alpha} = P(\,T \geq 0.5565) = 0.75\,, \tag{2.9}$$

which is a value typical for the null hypothesis. It is concluded on p. 38 of [2] on the basis of an outlier testing procedure that under the normality assumptions the smallest observation -1.4 is not discordant at the 5 per cent level. This can be demonstrated also by means of another tool. Indeed, when the data are processed by the Shapiro-Wilk test rule for the sake of testing the hypothesis that they come from a normal distribution, then one obtains that in this case the test statistic (cf. the pages 602, 603 and Table 6 on p. 605 of the section 3 of [11])

$$SW = \frac{\left(\sum_{i=1}^{m} a(i,n)\big(x^{(n+1-i)} - x^{(i)}\big)\right)^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 0.9398 > 0.881\,,$$

and the null hypothesis of normality is not rejected at the 5 per cent level. Since the small values of SW are critical, the infimum of the significance levels for which the Shapiro-Wilk test rejects the normality hypothesis is $P(\,SW \leq 0.9398)$. Simulations based on N=50 000 trials show that in this case under the normality assumptions the level attained

$$\hat{\alpha} = P(\,SW \leq 0.9398) = 0.37\,.$$

Though it is in a fair coincidence with the normality assumptions, it is still considerably less than the value of (2.9). One therefore may conclude that the sample comes from a Cauchy distribution.

The values of the probability of rejection the null hypothesis H_0 that the random sample $x_1, \dots, x_n$ comes from a Cauchy distribution, based on $\alpha = 0.05$ upper tail percentage points of the test statistics concerned (cf. Tables 2 and 3) and obtained from $N = 2000$ simulations for each particular test statistic, sample size n and alternative distribution, are given in the following Table 4, where L denotes the logistic, N the normal, U the uniform, B the beta and $StD(\nu)$ the Student's t distribution with ν degrees of freedom. Since in the location and scale parameter families the distributions of the underlying test statistics do not depend on the parameter, the logistic, normal and uniform distribution are mentioned without a reference to their parameter.

Table 4 suggests that the test (2.8) distinguishes many alternatives from the Cauchy distribution with a good performance, which one can assign to the fact that the alternatives concerned and the Cauchy distribution have tails of different order.

The construction of the test (2.8) is based on behaviour of the extreme order statistics. The referee pointed out that this leads to the situation that the asymptotic performance of this test will not be good for the alternatives with the same order of convergence of tails to 0 (as $|x| \to \infty$) as in the Cauchy distribution case, and the Kolmogorov-Smirnov test statistic will asymptotically distinguish such alternatives better. To demonstrate this property we consider here the alternatives, defined by means of the densities having the uniform middle part and the tails from the Cauchy distribution, i.e., the densities are defined by the formula

$$r(x, M) = \begin{cases} \dfrac{1}{\pi(1+x^2)} & |x| > M\,, \\ \\ \dfrac{\arctan(M)}{\pi M} & |x| \leq M\,, \end{cases} \tag{2.10}$$

where $M > 0$ is a parameter. Even though alternatives of this kind do not appear in lists of probabilities which are commonly used in statistical or probabilistic considerations (eg., the distributions studied in [6] and [7]), it is useful to consider them as

n	14					16				
	Q	KS	A^2	W^2	T	Q	KS	A^2	W^2	T
χ_1^2	0.75	0.56	0.32	0.32	0.28	0.87	0.66	0.43	0.42	0.46
χ_2^2	0.40	0.19	0.09	0.09	0.08	0.56	0.25	0.14	0.13	0.17
χ_5^2	0.14	0.03	0.02	0.02	0.02	0.25	0.04	0.03	0.03	0.04
L	0.05	0.01	0.01	0.01	0.01	0.07	0.01	0.01	0.01	0.02
N	0.06	0.01	0.01	0.01	0.01	0.10	0.00	0.00	0.00	0.01
U	0.32	0.04	0.03	0.03	0.03	0.50	0.05	0.05	0.03	0.08
$B(1,2)$	0.29	0.05	0.03	0.03	0.03	0.43	0.07	0.05	0.05	0.07
$B(2,2)$	0.14	0.01	0.01	0.01	0.01	0.24	0.01	0.02	0.02	0.04
$StD(2)$	0.03	0.01	0.00	0.01	0.01	0.03	0.01	0.01	0.01	0.02
$StD(5)$	0.05	0.01	0.01	0.01	0.01	0.07	0.01	0.01	0.01	0.02

n	18					20				
	Q	KS	A^2	W^2	T	Q	KS	A^2	W^2	T
χ_1^2	0.95	0.77	0.51	0.48	0.63	0.98	0.85	0.62	0.58	0.60
χ_2^2	0.70	0.33	0.19	0.17	0.27	0.84	0.41	0.26	0.22	0.22
χ_5^2	0.37	0.06	0.05	0.04	0.08	0.52	0.09	0.08	0.07	0.06
L	0.10	0.01	0.01	0.01	0.03	0.17	0.01	0.02	0.02	0.03
N	0.16	0.01	0.01	0.01	0.04	0.24	0.01	0.02	0.01	0.03
U	0.67	0.06	0.09	0.05	0.17	0.82	0.09	0.13	0.07	0.13
$B(1,2)$	0.60	0.12	0.10	0.08	0.16	0.77	0.15	0.14	0.10	0.10
$B(2,2)$	0.38	0.03	0.04	0.03	0.08	0.53	0.03	0.06	0.03	0.06
$StD(2)$	0.05	0.01	0.01	0.01	0.02	0.07	0.01	0.01	0.01	0.02
$StD(5)$	0.10	0.01	0.01	0.01	0.03	0.14	0.01	0.01	0.01	0.02

n	25					35				
	Q	KS	A^2	W^2	T	Q	KS	A^2	W^2	T
χ_1^2	1.00	0.95	0.81	0.75	0.79	1.00	1.00	0.98	0.96	0.97
χ_2^2	0.98	0.62	0.43	0.34	0.41	1.00	0.92	0.79	0.62	0.71
χ_5^2	0.79	0.16	0.17	0.12	0.18	0.99	0.41	0.43	0.26	0.39
L	0.32	0.02	0.03	0.02	0.05	0.67	0.04	0.11	0.06	0.14
N	0.48	0.02	0.05	0.04	0.08	0.87	0.05	0.18	0.08	0.21
U	0.97	0.19	0.31	0.15	0.28	1.00	0.51	0.72	0.41	0.57
$B(1,2)$	0.95	0.31	0.29	0.19	0.28	1.00	0.67	0.68	0.42	0.57
$B(2,2)$	0.84	0.06	0.15	0.08	0.15	1.00	0.16	0.45	0.19	0.36
$StD(2)$	0.10	0.02	0.02	0.01	0.03	0.20	0.02	0.03	0.03	0.06
$StD(5)$	0.29	0.02	0.03	0.03	0.05	0.57	0.03	0.09	0.05	0.13

Table 4. The power of the statistics (2.6), (1.7), (1.1), (1.2) and $T = Q(n, 0.1, 0.9)$ based on the median (1.8) and on the trigonometric scale estimate (1.6).

indicators of weak spots of the presented extreme order statistics test. In obtaining some numerical results in this respect, the following percentage points, obtained from $N = 50000$ simulations for each considered sample size, will be used.

n	30	50	150	300	500	1000
$q(0.05, n)$	9.3099	8.5822	7.9964	7.9798	8.0510	7.8879
$ks(0.05, n)$	1.0180	0.9608	0.9012	0.8871	0.8861	0.8764

Table 5. Five percent upper tail percentage points of the statistics A^2 and KS based on (1.8) and (1.6).

An outline of the behaviour of the power under validity of the mentioned alternatives is given in Table 6, containing the simulation estimates (based on $N = 2000$ trials) of the probabilities $P(\,Q > q(0.05, n)\,)$, $P(\,KS > ks(0.05, n)\,)$ of rejection the null hypothesis that the true distribution is a Cauchy distribution, when the sampling of size n is drawn from the alternatives (2.10) and when the rejection constants, corresponding to the significance level $\alpha = 0.05$, are those given in the previous table. The values 1, 2.4142 and 6.3138 of the parameter M are here chosen to obtain the distribution, having on the outer parts the total mass β of the Cauchy type (i.e, $\beta/2$ on each side), with the values $\beta = 0.5$, 0.25 and 0.1, respectively.

n	$M = 1$		$M = 2.4142$		$M = 6.3138$	
	Q	KS	Q	KS	Q	KS
30	0.06	0.05	0.16	0.08	0.53	0.16
50	0.06	0.05	0.18	0.18	0.56	0.54
150	0.06	0.09	0.17	0.71	0.58	1.00
300	0.07	0.16	0.20	0.98	0.59	1.00
500	0.07	0.23	0.18	1.00	0.56	1.00
1000	0.06	0.47	0.18	1.00	0.57	1.00

Table 6. The power of the statistics (2.6) and (1.7) based on the estimators (1.8) and (1.6).

The results of simulations from Table 6 suggest that the test (2.8) will be asymptotically insensitive against the alternatives (2.10). Still, these simulations also show, that for $n < 50$ the test (2.8) is more powerful under (2.10) than the Kolmogorov-Smirnov test.

On the whole one may conclude, that the results of simulations testify in favor of the test (2.8) based on the statistic $Q = Q(n, 1/(n+1), n/(n+1))$ defined by (2.6), (2.7). Indeed, as it can be seen from Table 4, the statistic Q possesses for the alternatives considered ibidem the power fairly close to 1 for sample size $n = 35$, with the exception of the logistic distribution and the Student's distribution having degrees of freedom not too much different from 1 (the Student's distributions include the Cauchy $C(0, 1)$ distribution, because $C(0, 1) = StD(1)$). In each case considered in Table 4 the test based on Q performs clearly better than its mentioned competitors. It is therefore recommendable to test goodness-of-fit for the Cauchy distribution against the alternatives with different order of the tail convergence by means of the test (2.8) based on the extreme order quantile statistic $Q(n, 1/(n+1), n/(n+1))$.

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